Phase-space cascades in *(simulations of)* plasma turbulence

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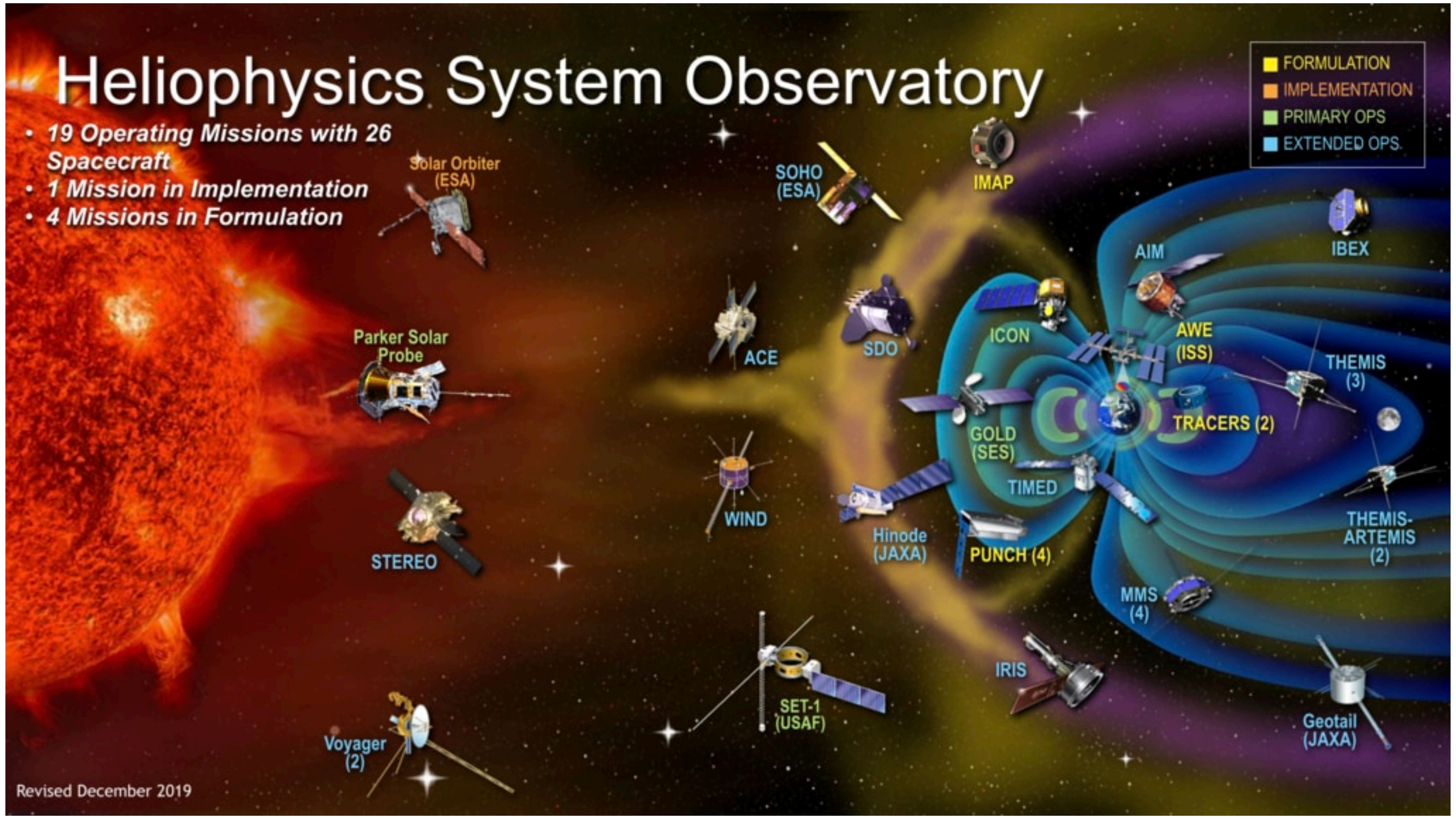
Thanks:
M. W. Kunz (Princeton University)
F. Califano (Università di Pisa)

Interconnections between the physics of plasmas and self-gravitating systems — KITP, July 1st, 2024

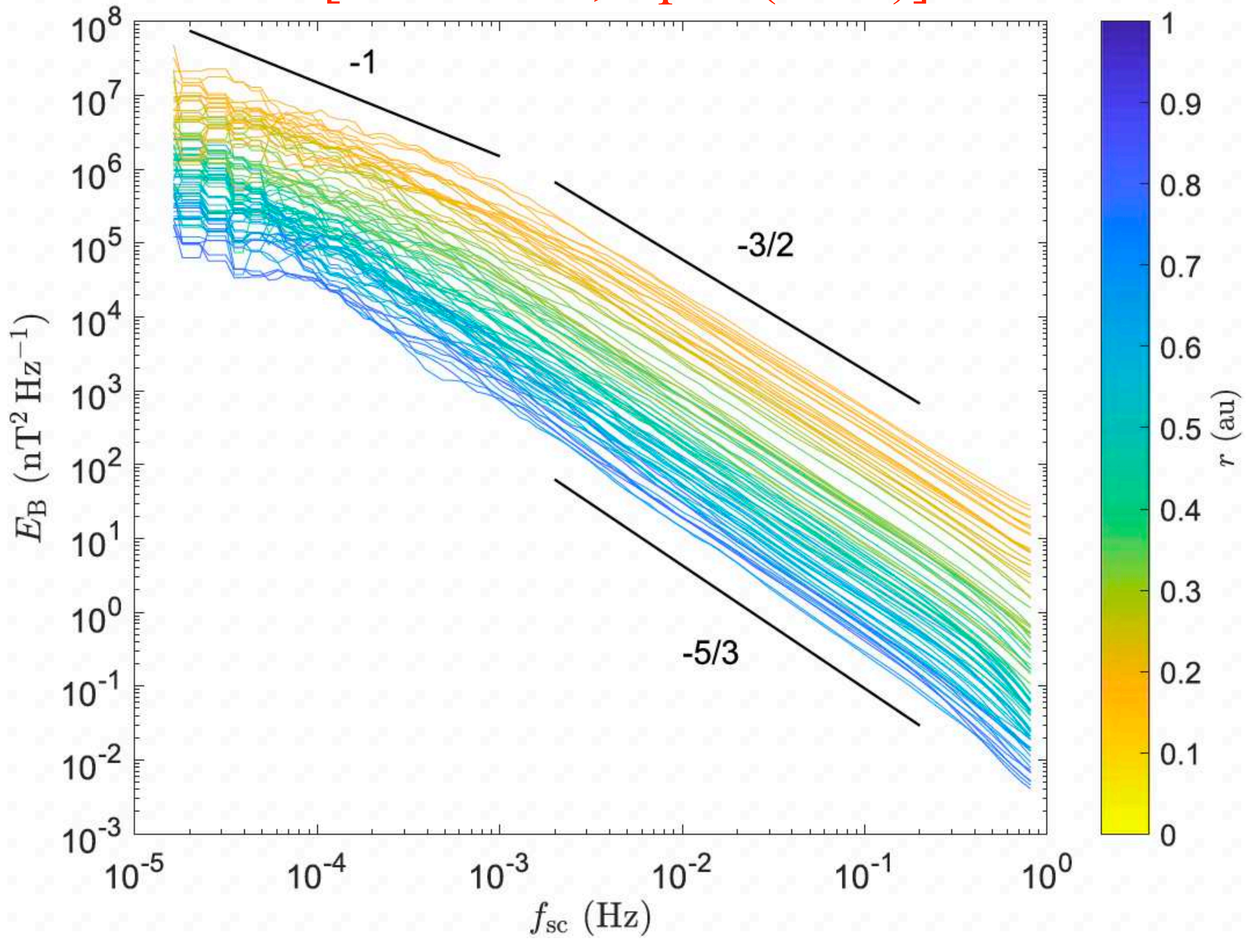
The solar wind as plasma-turbulence laboratory

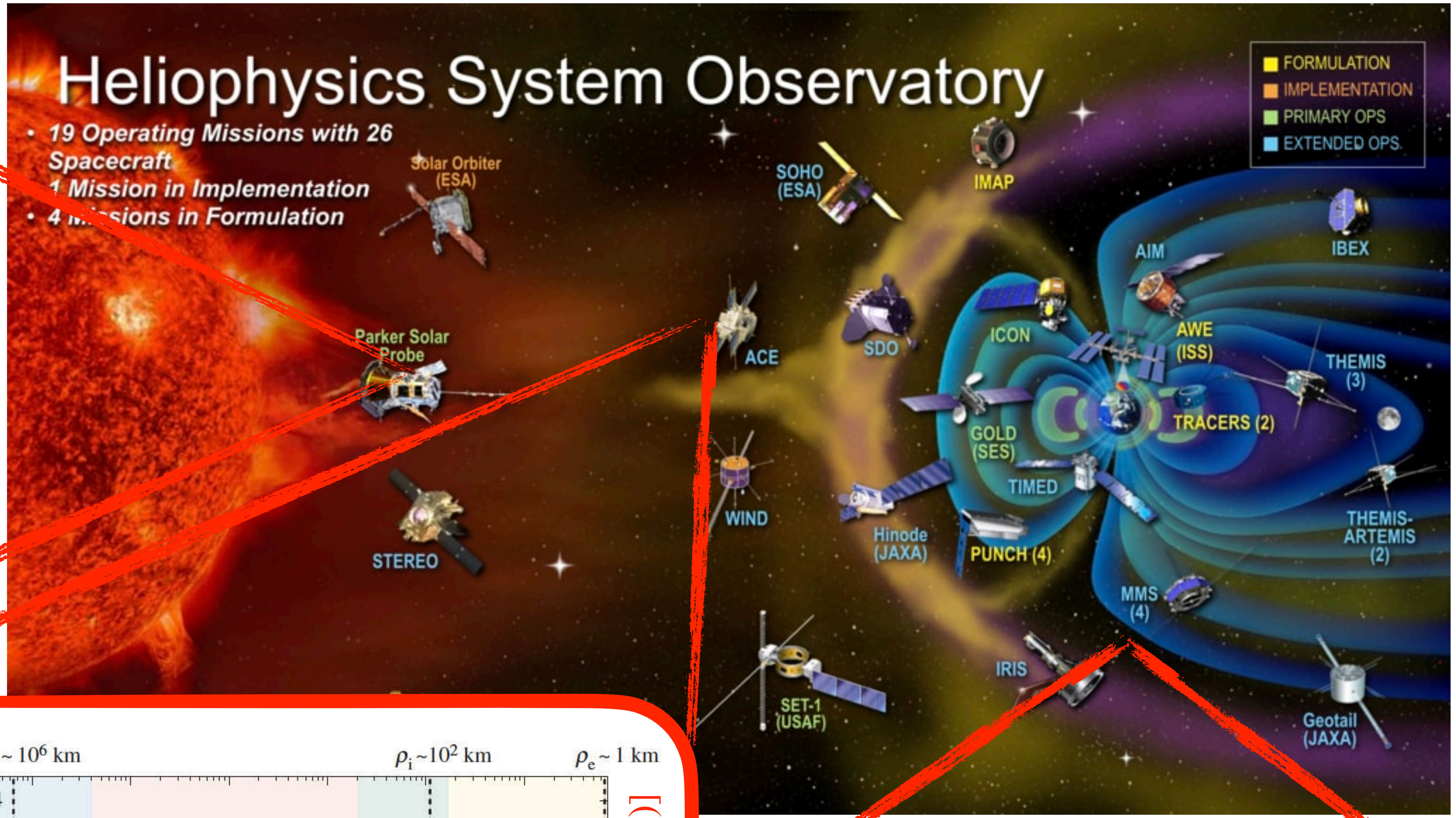


[credit: NASA]

The solar wind as plasma-turbulence laboratory

[Chen et al., ApJS (2020)]



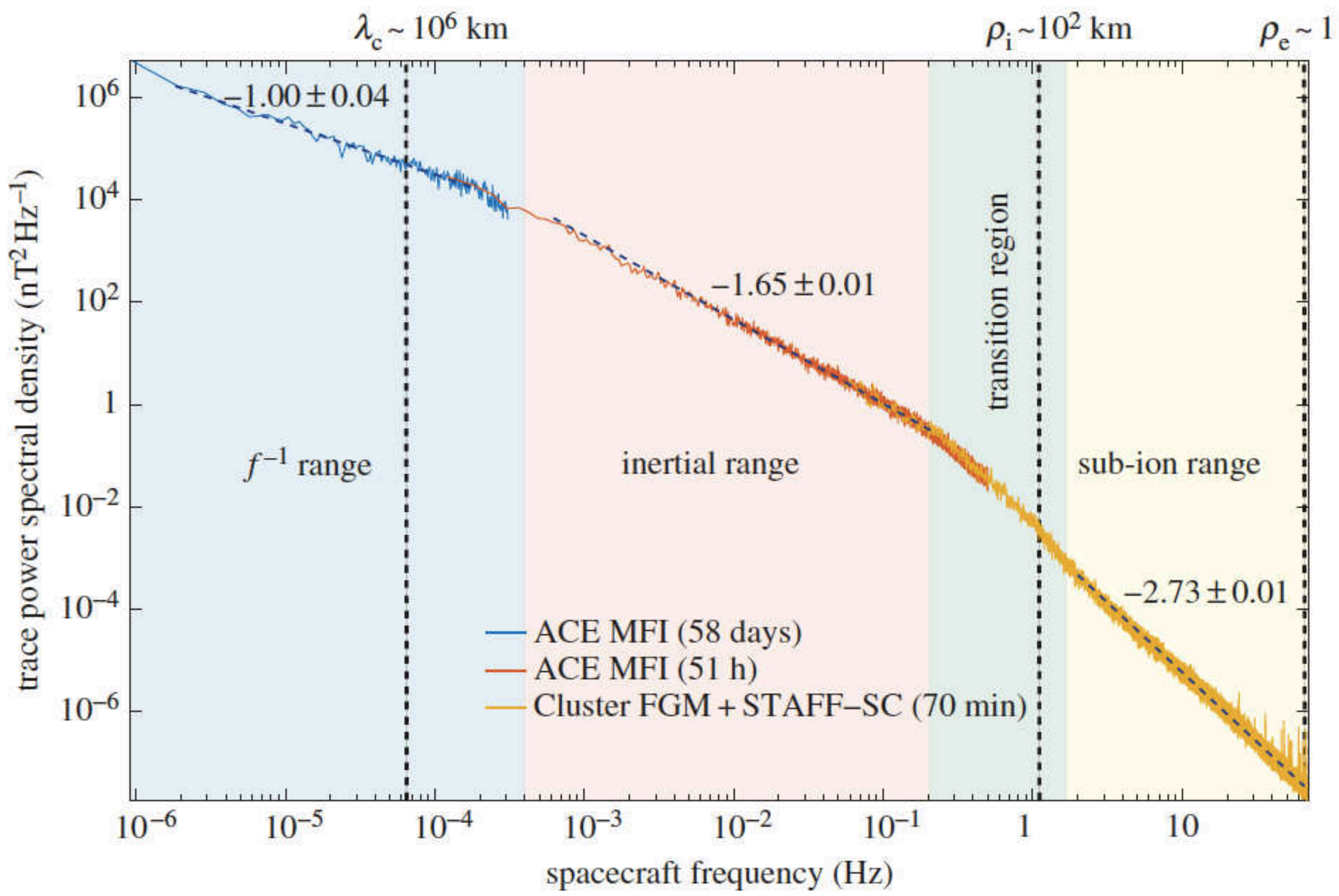


[credit: NASA]

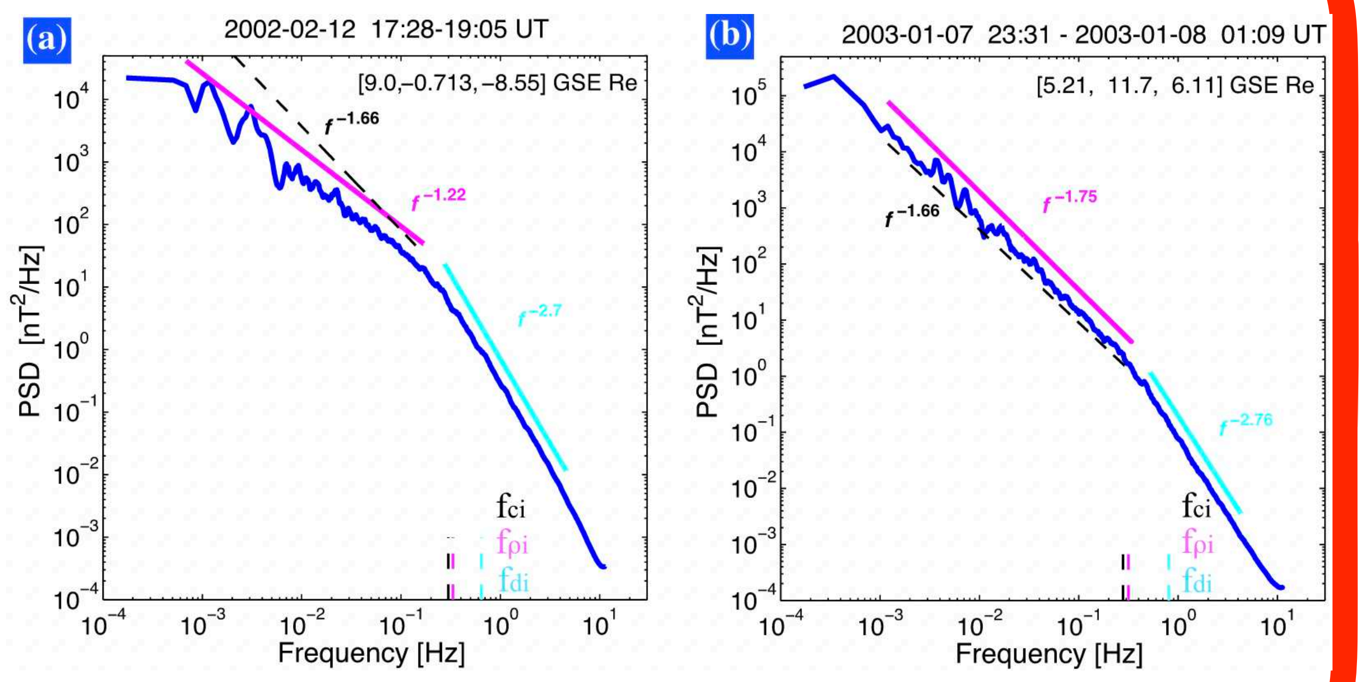
A huge amount of detailed **in-situ data** of solar-wind turbulence

☞ some reviews:

Bruno & Carbone, LRSP (2013)
Chen, JPP (2016)
Sahraoui et al., RvMPP (2020)



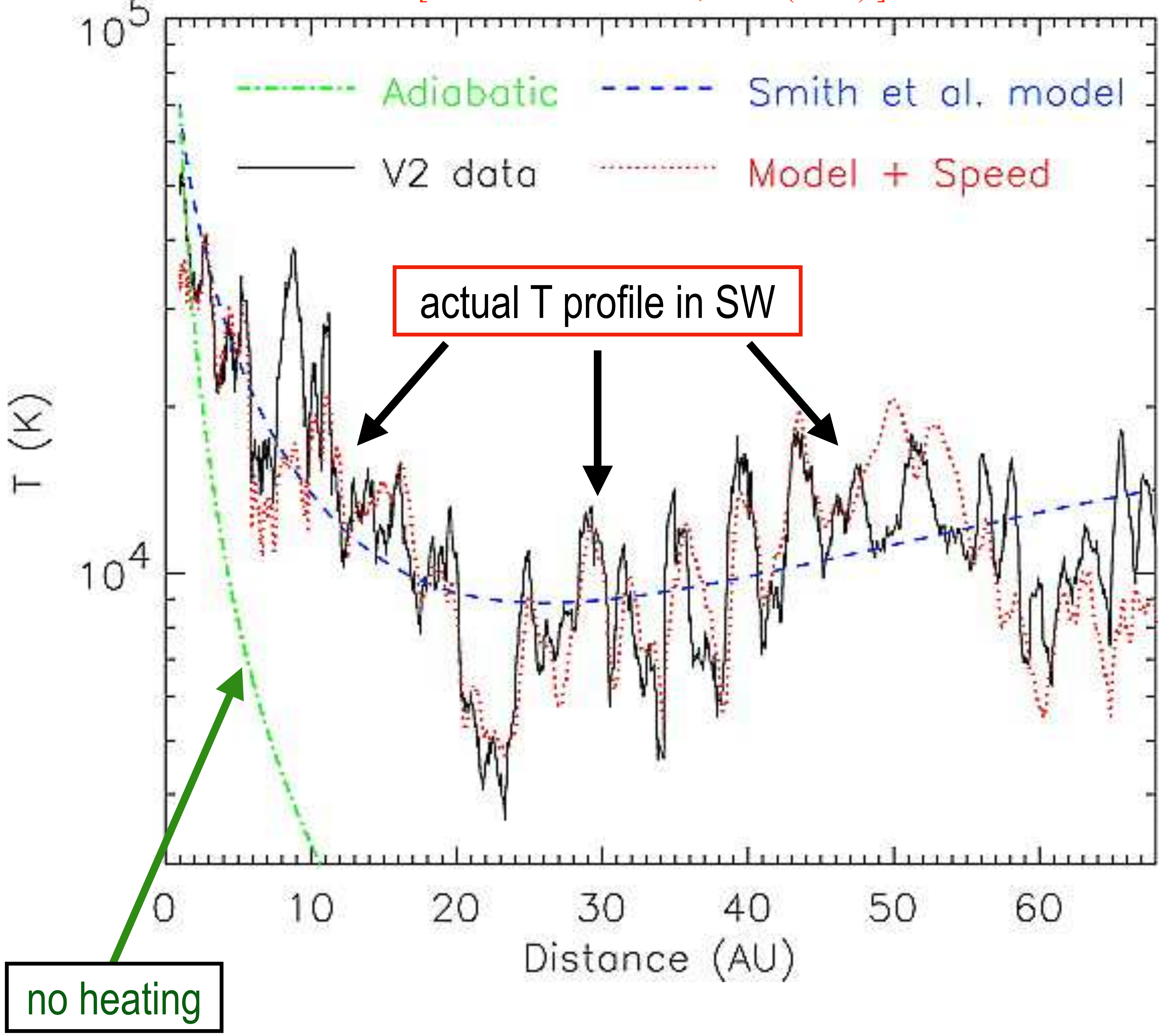
[Kiyani et al., Phil Trans A (2015)]



[Huang et al., ApJL 2017]

Turbulent heating in the solar wind

[Richardson & Smith, GRL (2003)]



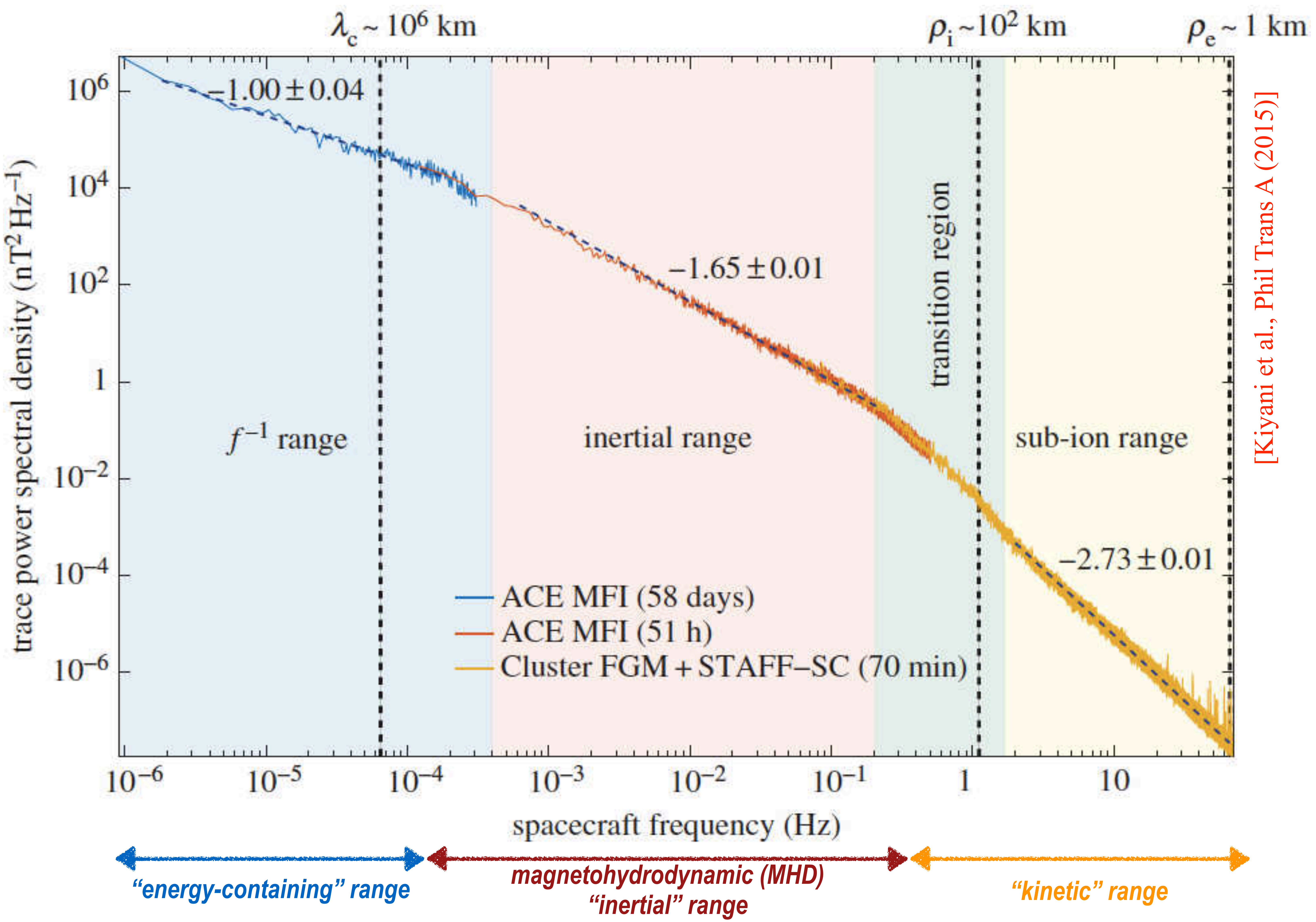
- Solar-wind (SW) temperature profile vs distance: **decreases slower than** it should for a simple **adiabatic expansion**

- ***Dissipation of turbulent fluctuations*** can account for the observed ***extra heating***

- SW is very **weakly collisional** → ***kinetic effects*** must play a relevant role in ***turbulent dissipation***

- (**preferential heating** of ions **perpendicular to *B***)

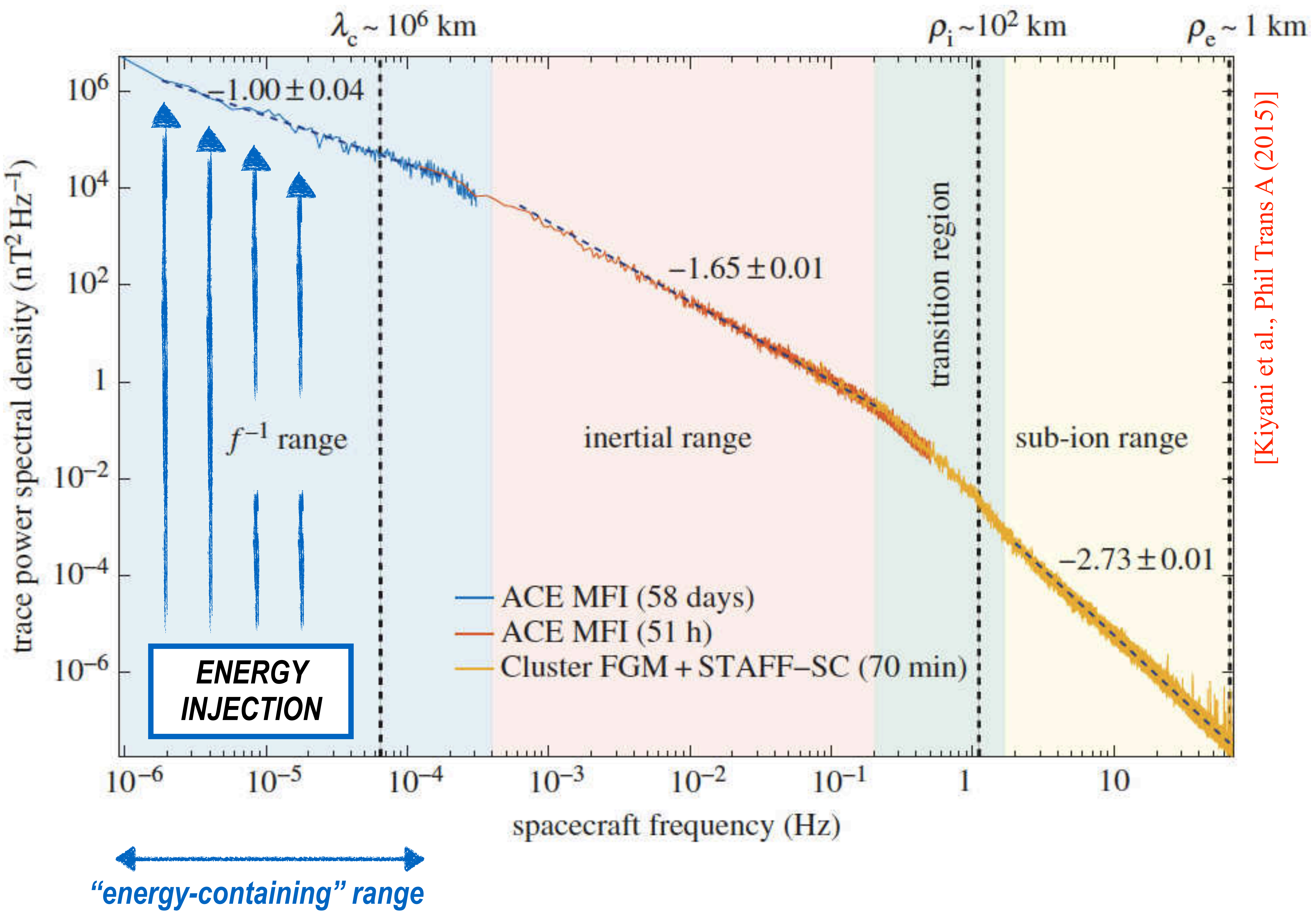
Turbulent cascade in the solar wind



Taylor hypothesis:

$\omega \approx \mathbf{k} \cdot \mathbf{V}_{SW}$

Turbulent cascade in the solar wind



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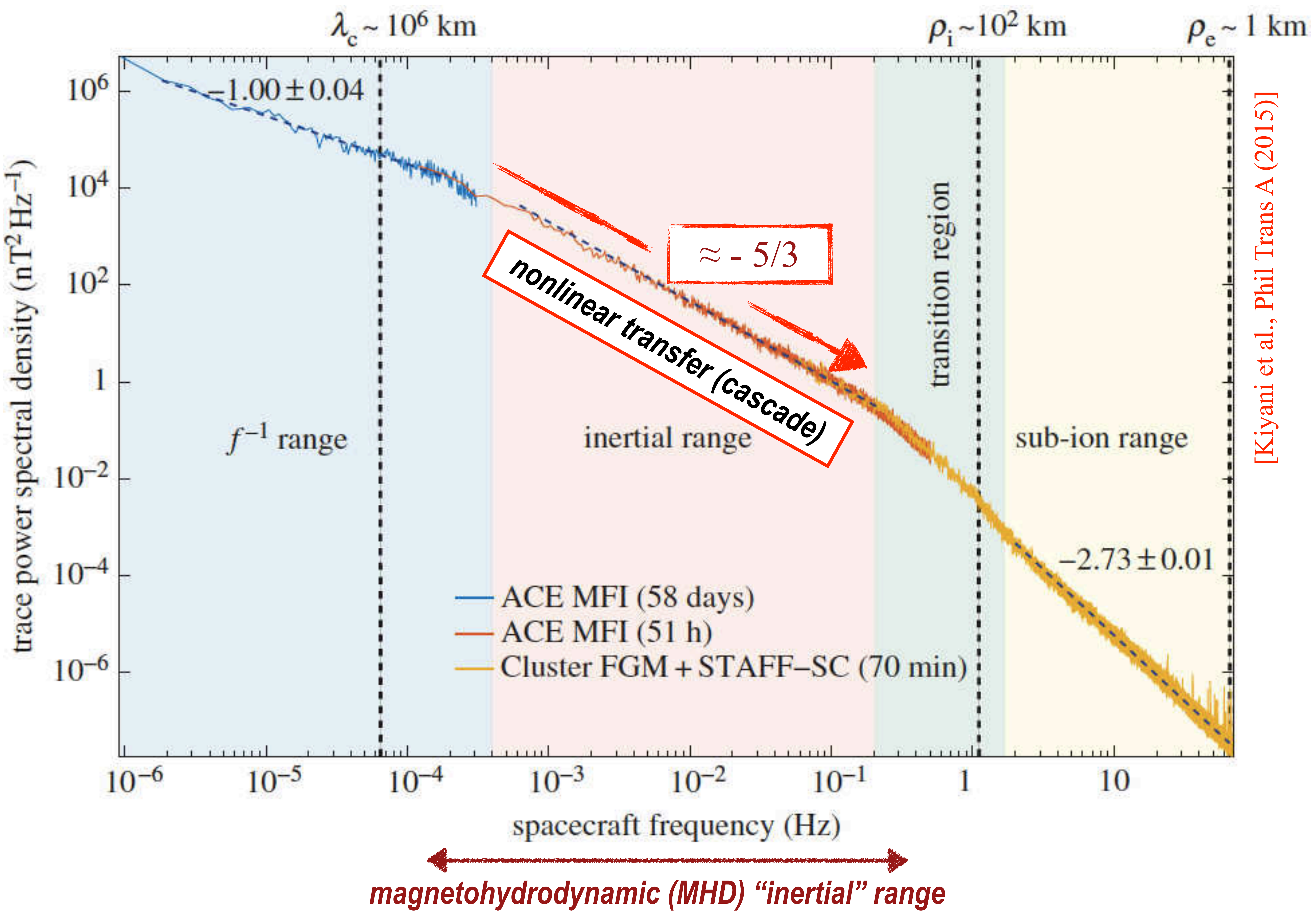

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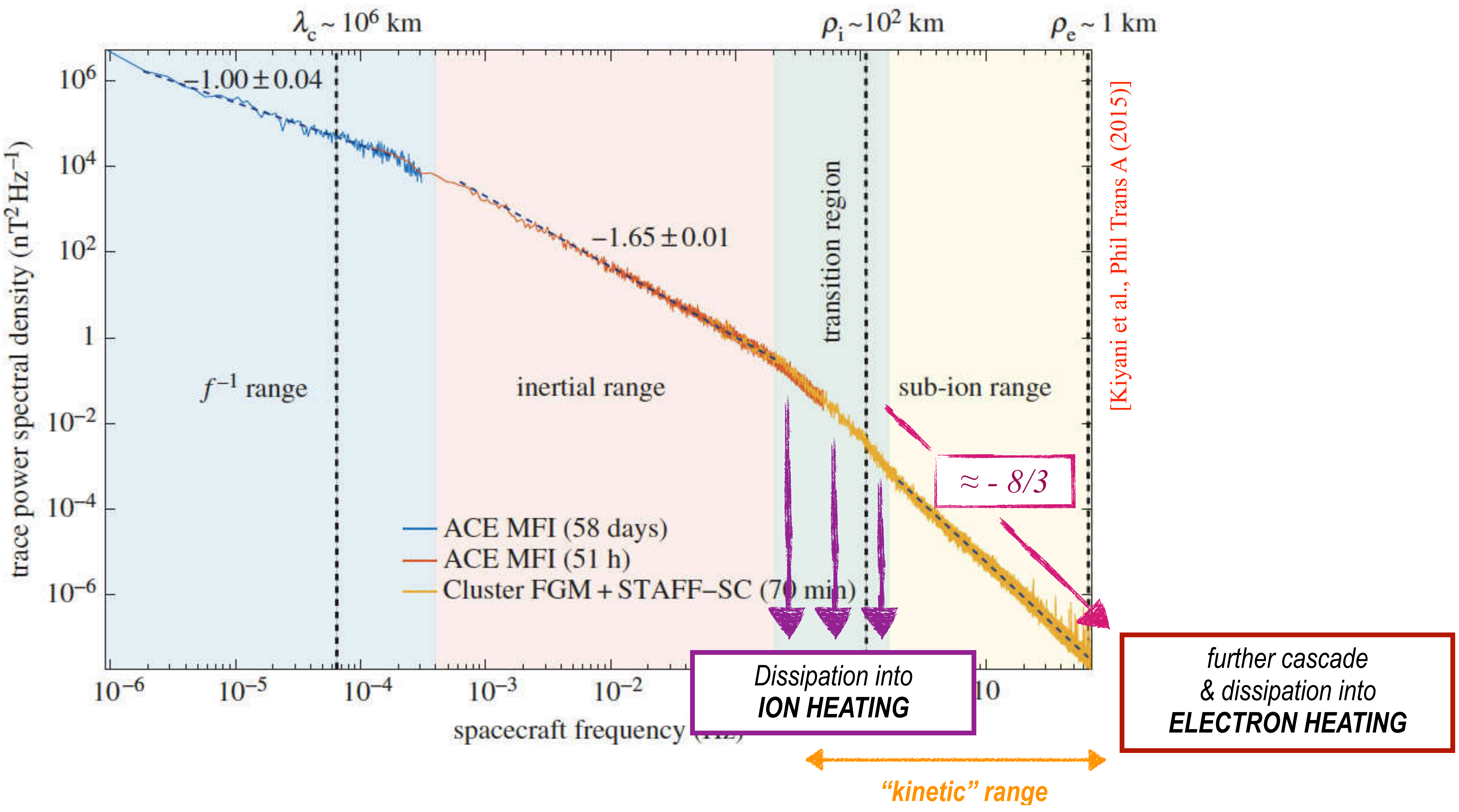

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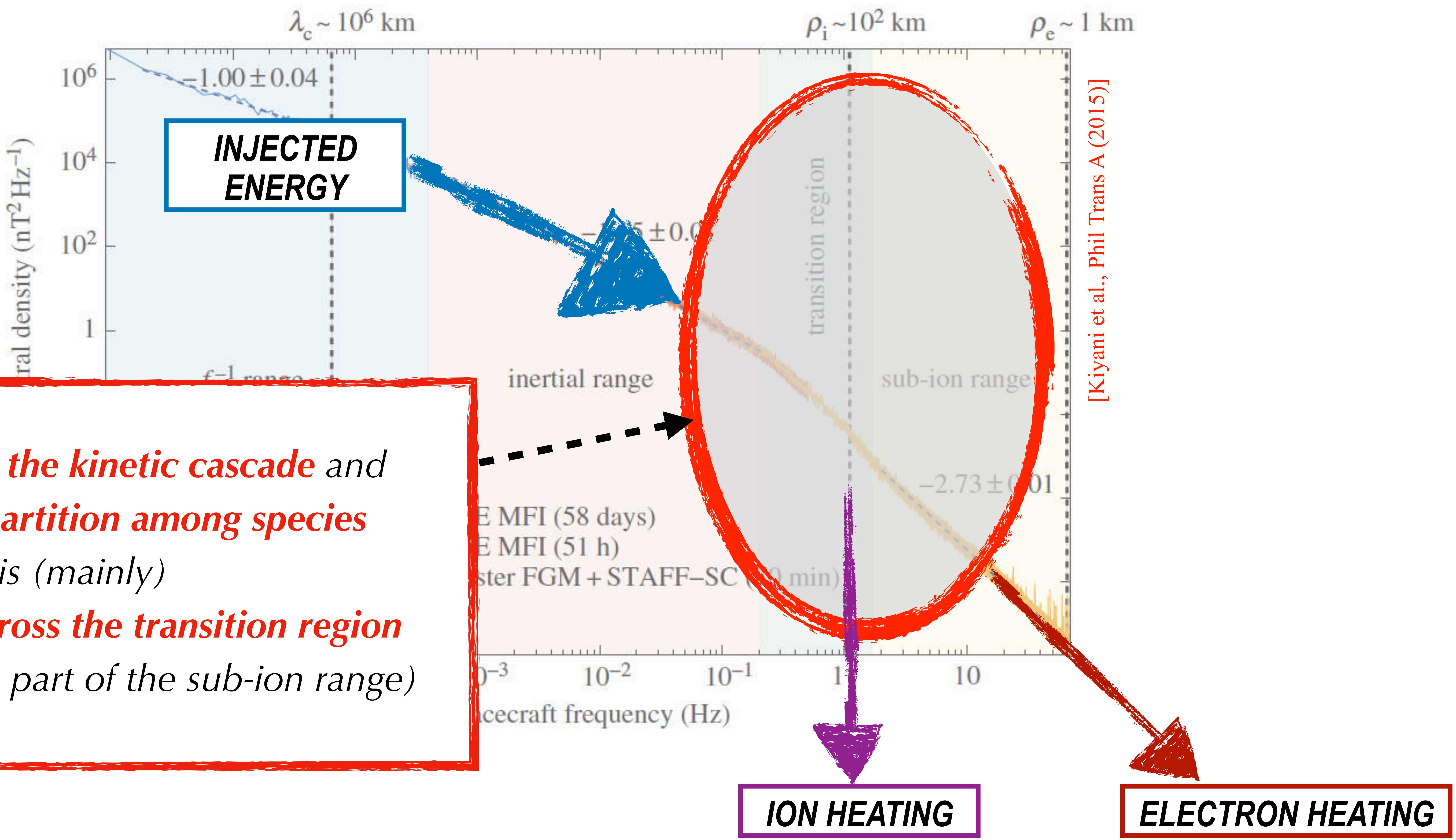

The ***nature of the kinetic cascade*** *and the* ***energy partition among species*** *is (mainly)* ***determined across the transition region*** *(and perhaps in part of the sub-ion range)*

(some) ion-heating mechanisms

☞ Resonant mechanisms:

➢ **Landau damping** (*parallel* heating)

➥ heating model(s): Howes MNRAS (2010), Kawazura+ PNAS (2019)

➢ **Ion-cyclotron damping** (*perpendicular* heating)

➥ heating model(s): Arunasalam PRL (1976), Cranmer ApJS (2014)

☞ Non-resonant mechanism:

➢ **Stochastic heating** (*perpendicular* heating)

➥ heating model(s): Chandran+ ApJ (2010), Cerri+ ApJ (2021)

(some) ion-heating mechanisms

PLASMAS | STELLAR SYSTEMS

☞ Resonant mechanisms:

➢ **Landau damping** (*parallel* heating) → Corotation Resonance (?)

➥ heating model(s): Howes MNRAS (2010), Kawazura+ PNAS (2019)

➢ **Ion-cyclotron damping** (*perpendicular* heating) → Lindblad Resonances (?)

➥ heating model(s): Arunasalam PRL (1976), Cranmer ApJS (2014)

☞ Non-resonant mechanism:

➢ **Stochastic heating** (*perpendicular* heating) → Radial Migration (?)

➥ heating model(s): Chandran+ ApJ (2010), Cerri+ ApJ (2021)

Phase-space dynamics

☞ What is the behavior of δf in phase space? (and how to conveniently represent it?)

$$\delta f(\boldsymbol{x}, \boldsymbol{v}) \doteq f(\boldsymbol{x}, \boldsymbol{v}) - F_{\mathrm{M}}(\boldsymbol{x}, \boldsymbol{v})$$

Phase-space dynamics

☞ What is the behavior of δf in phase space? (and how to conveniently represent it?)

linear ("parallel") phase mixing

non-linear ("perpendicular") phase mixing

Phase-space dynamics

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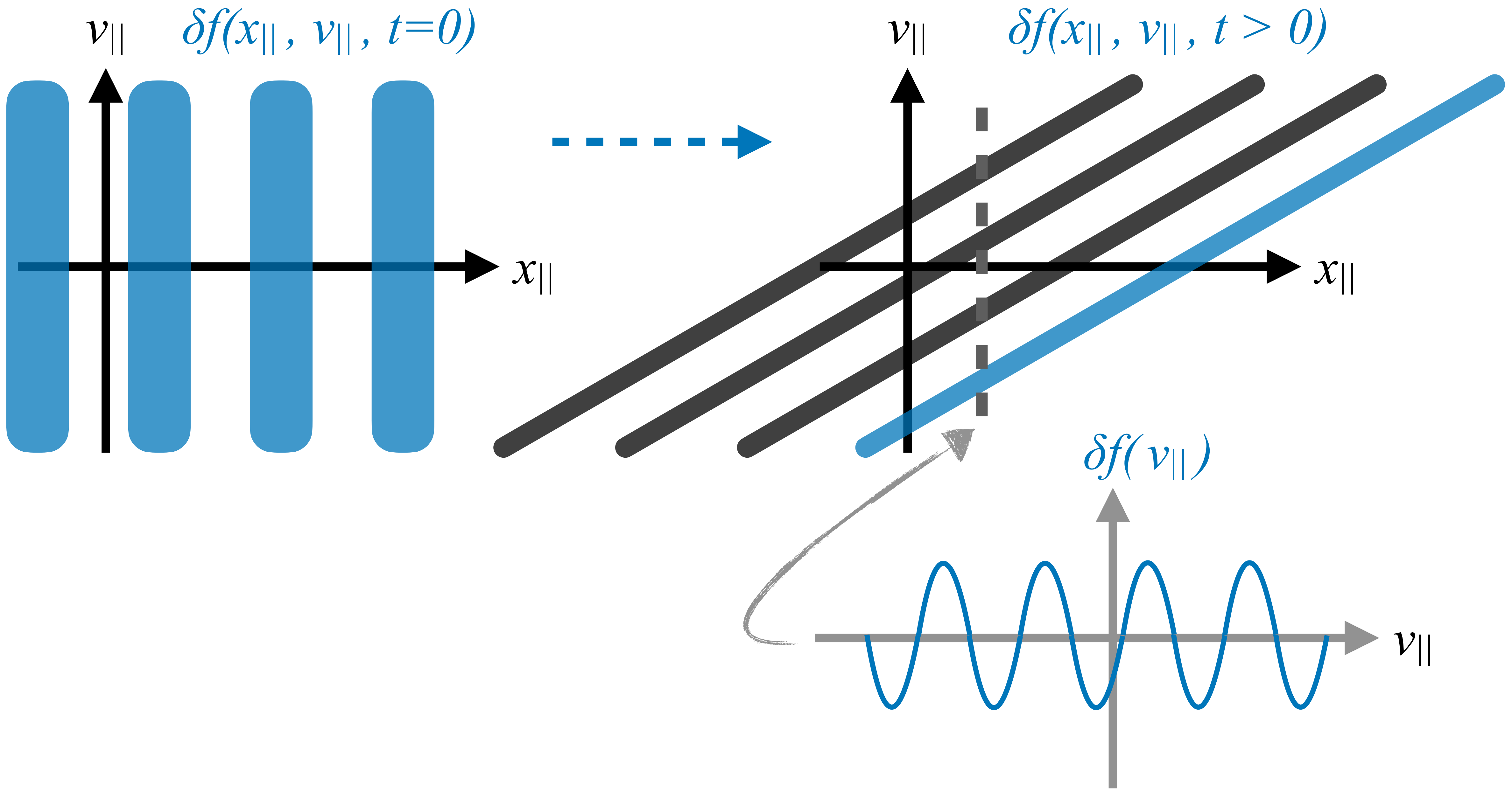
linear ("parallel") phase mixing

- due to ballistic evolution of δf along $\boldsymbol{B}$ ("parallel"):

$$\delta f \sim exp(-ik_{||}v_{||}t)$$



- slower than its nonlinear counterpart at scales below the ion gyro-radius ρ_i (in strong turbulence)
  → dominant only at $k_\perp \rho_i < 1$

non-linear ("perpendicular") phase mixing

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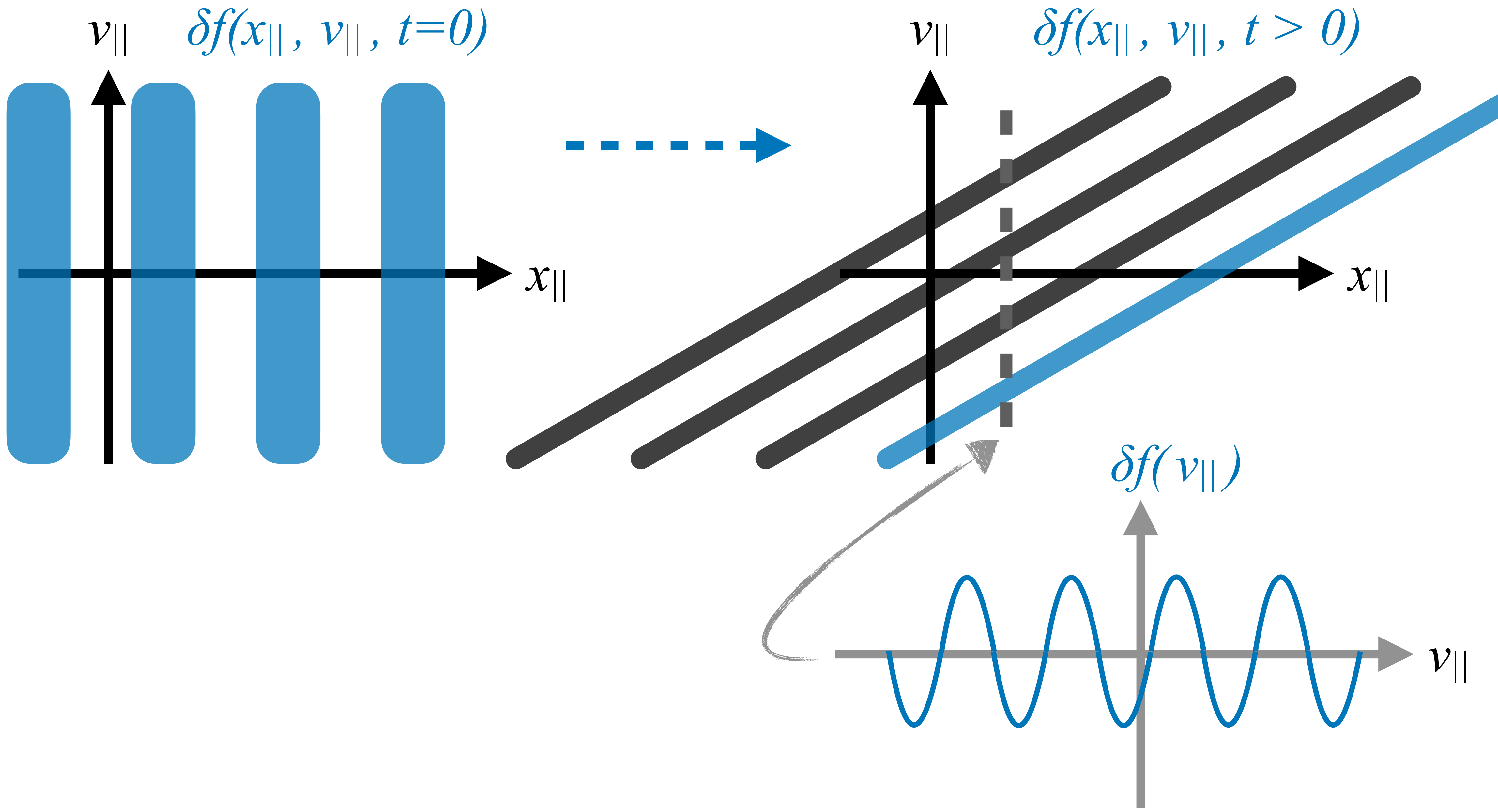

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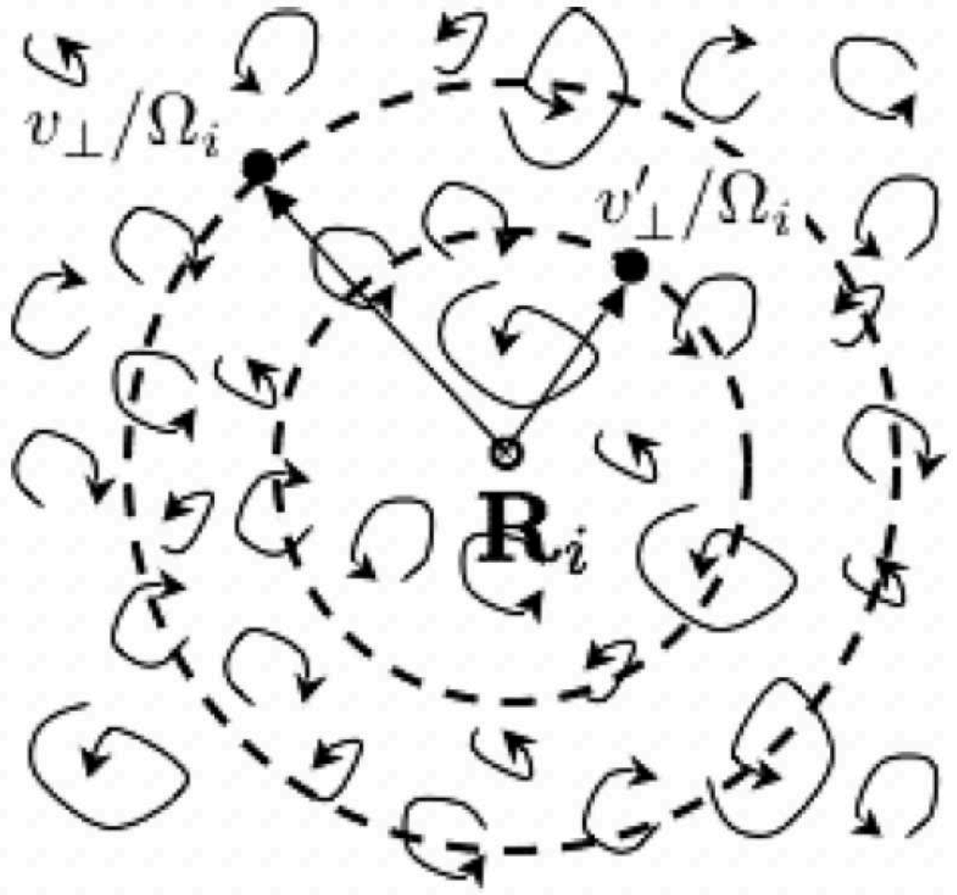
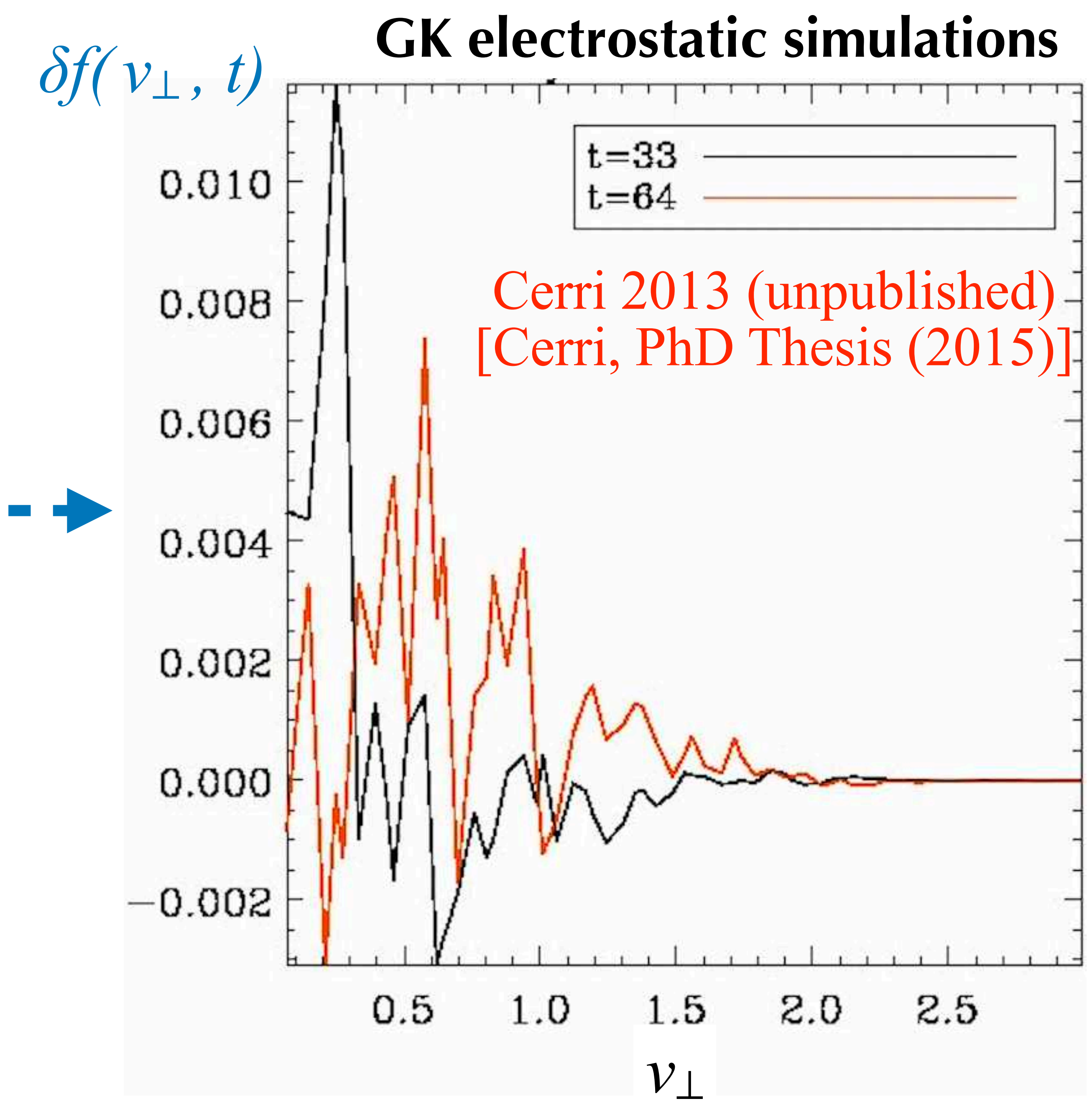
non-linear ("perpendicular") phase mixing

- due to de-correlation of "$v_\perp$-slices" of δf from transport over de-correlated $k_\perp$-fluctuations in the plane perp. to $\boldsymbol{B}$:



[Schekochihin et al., PPCF (2008)]



- occurring only at scales below the ion gyro-radius ($k_\perp \rho_i \gg 1$)

- faster than its linear counterpart (in strong turbulence)

Phase-space dynamics

☞ What is the behavior of δf in phase space? (**<u>and how to conveniently represent it</u>**?)

Phase-space dynamics

☞ What is the behavior of δf in phase space? (and how to conveniently represent it?)

▶ ***Hermite representation:***

$$\delta f_{\boldsymbol{m}} \doteq \langle \delta f_{\boldsymbol{m}}(\boldsymbol{x}) \rangle = \left\langle \int_{-\infty}^{+\infty} \delta f(\boldsymbol{x}, \boldsymbol{v})\, \psi_{\boldsymbol{m}}(\boldsymbol{v})\, \mathrm{d}^3\boldsymbol{v} \right\rangle$$

- ⟨…⟩ = spatial average

- basis functions:

$$\psi_m(v) = \frac{H_m(v/v_{\rm th})}{\sqrt{2^m\, m!\, \sqrt{\pi} v_{\rm th}}}\, e^{-v^2/(2v_{\rm th}^2)}$$

$$\int_{-\infty}^{+\infty} \psi_n(v)\, \psi_m(v)\, \mathrm{d}v = \delta_{nm}$$

- H_m = "physicists' Hermite polynomials"

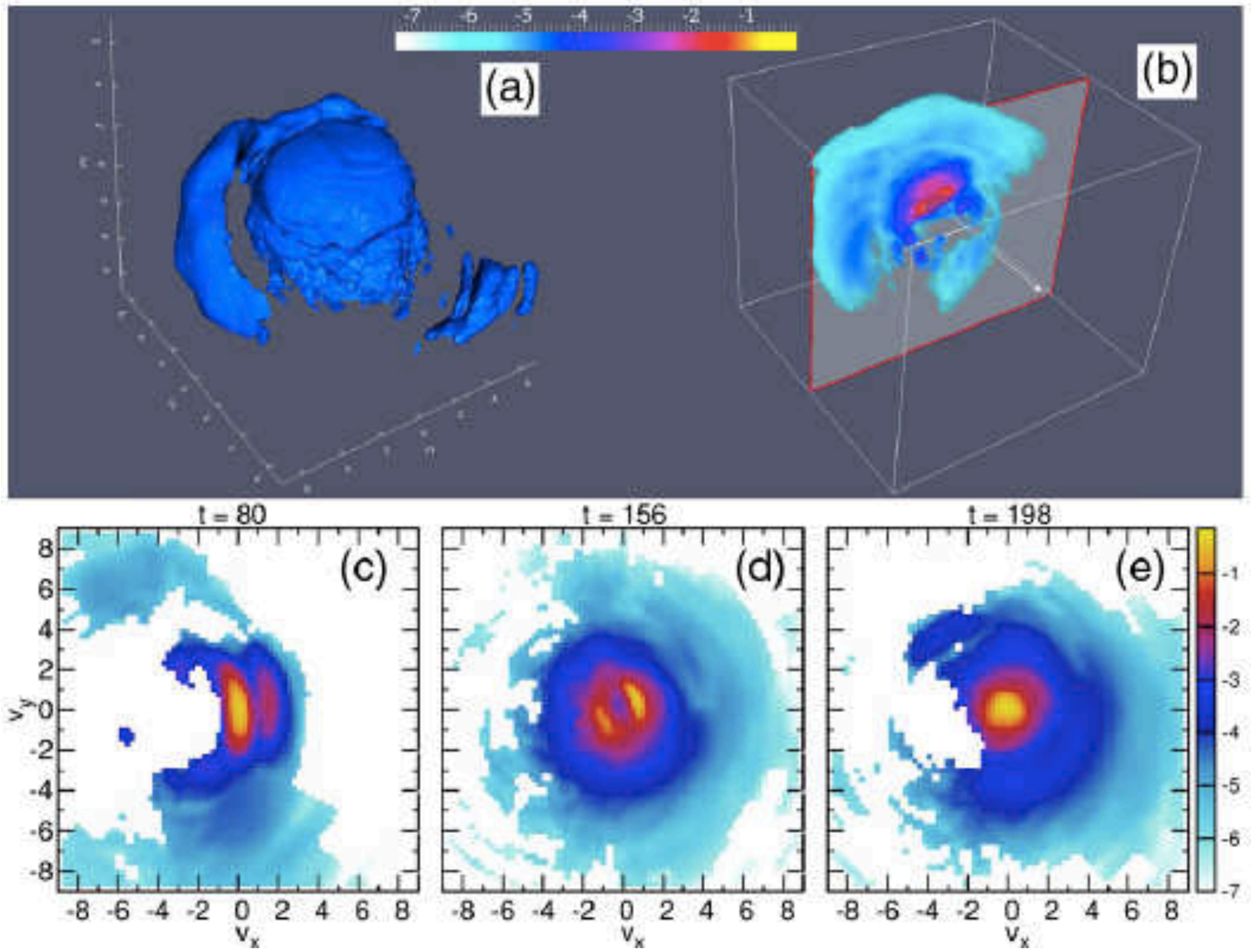
$$H_m(x) \doteq (-1)^m\, e^{x^2} \frac{\mathrm{d}^m}{\mathrm{d}x^m}\, e^{-x^2}$$

Phase-space dynamics

☞ What is the behavior of δf in phase space? (and how to conveniently represent it?)

Hermite representation — δf_{ion} from in-situ data (*MMS in the turbulent Earth's magnetosheath; $\delta B/B \sim 1.5$*)

$\delta f(\boldsymbol{v})$



[Servidio et al., PRL (2017)]

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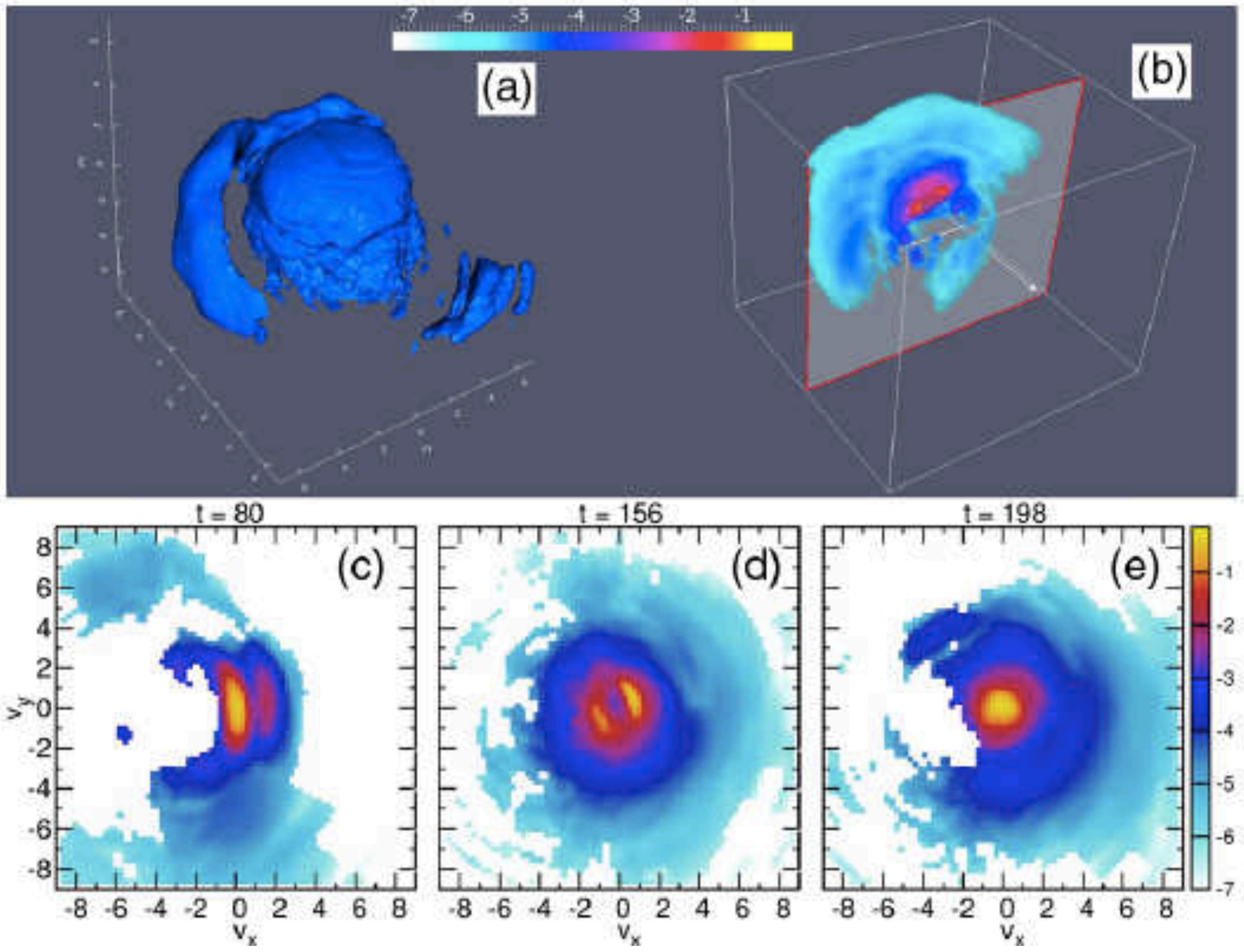
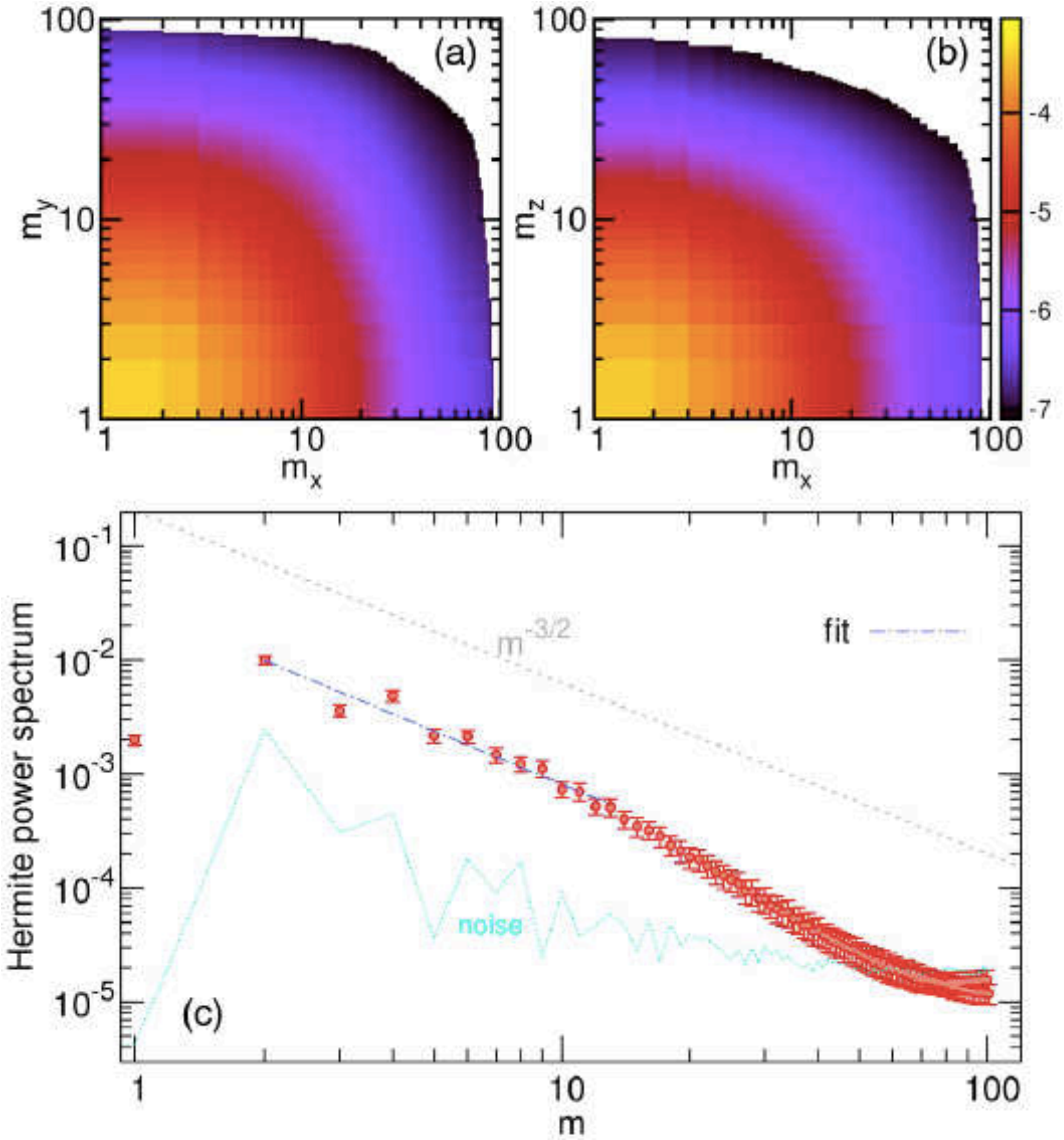
$\delta f(\mathbf{v})$ - - - - - - - - ▶ δf_m





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Phase-space dynamics

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▶ ***Hermite representation — what to expect for δf spectrum from different processes?***

Phase-space dynamics

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Hermite representation — what to expect for δf spectrum from different processes?

➢ ***linear phase mixing*** (*in parallel-to-**B** m-space*) ⟶ $\sim m_{\|}^{-1/2}$

[Watanabe & Sugama, PoP (2004); Zocco & Schekochihin, PoP (2011); Kanekar et al., JPP (2015); Adkins & Schekochihin, JPP (2018); …]

➢ ***non-linear phase mixing*** (*in field-perpendicular m-space*)

- ***classic GK theory in kinetic-Alfvén-wave (KAW) turbulence*** ⟶ $\sim m_{\perp}^{-7/6}$

  [Schekochihin et al., ApJS (2009); Cerri et al., ApJ (2018)]

- ***intermittency corrected KAW cascade*** ⟶ $\sim m_{\perp}^{-4/3}$

  [Cerri et al., ApJ (2018)]

- ***constant-anisotropy sub-ion-scale regime*** ⟶ $\sim m_{\perp}^{-3/2}$

  [Cerri et al., ApJ (2018)]

➢ ***hydro-like agruments on Vlasov eq.*** (*$\boldsymbol{v}$-space only, no $\boldsymbol{k}$-dep. of flucts, isotropic in m*) ⟶ $\sim m^{-3/2}$ or $\sim m^{-2}$

[Servidio et al., PRL (2017)]

Phase-space dynamics

☞ What is the behavior of δf in phase space? (and how to conveniently represent it?)

▶ ***Fourier-Hermite representation:***

$$\delta f_{\boldsymbol{k},\boldsymbol{m}} \doteq \frac{1}{(2\pi)^{3/2}} \int\int \delta f(\boldsymbol{x},\boldsymbol{v})\, e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}\, \psi_{\boldsymbol{m}}(\boldsymbol{v})\, \mathrm{d}^3\boldsymbol{x}\, \mathrm{d}^3\boldsymbol{v}$$

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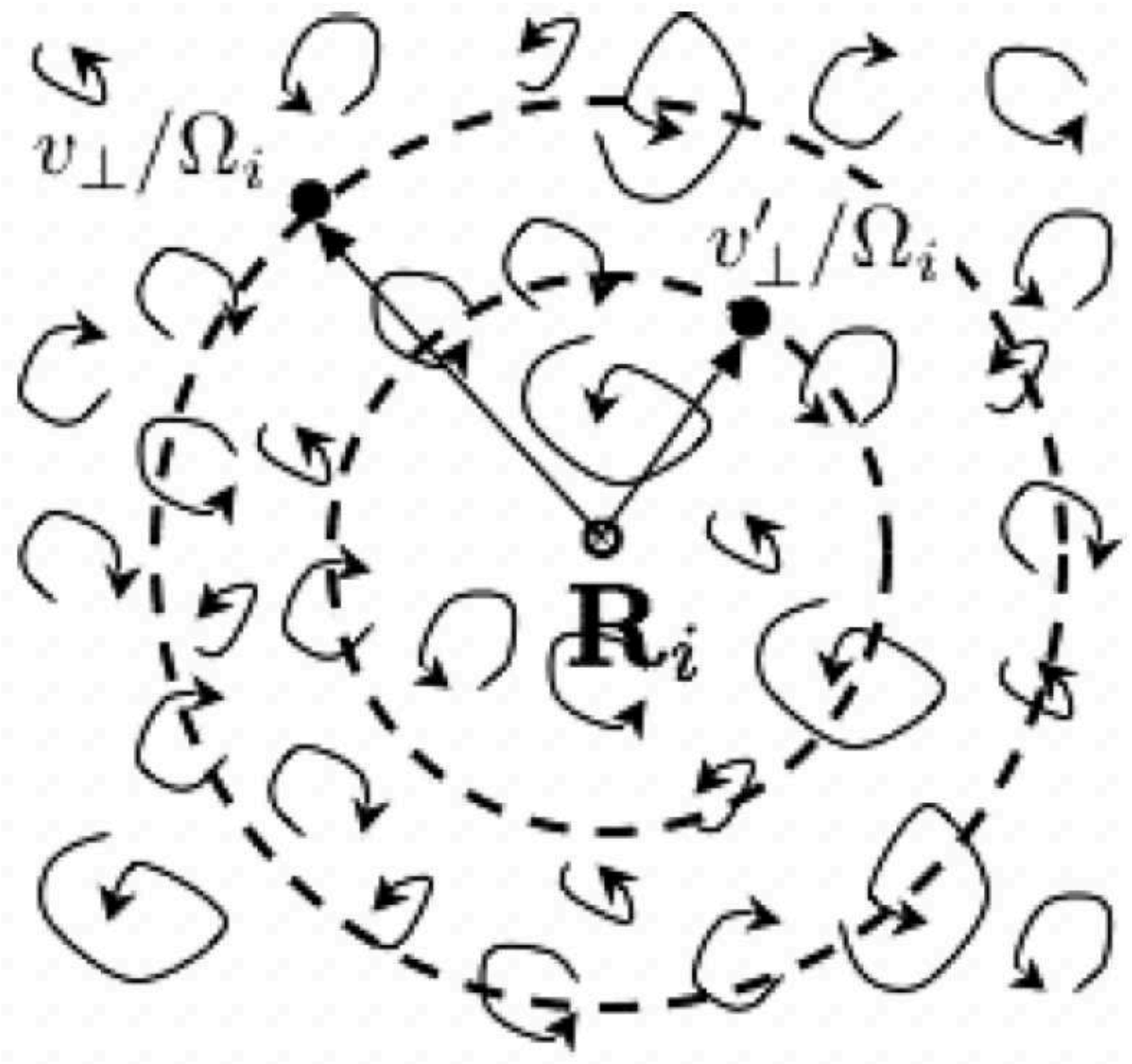

$$\frac{\delta v_\perp}{v_{\mathrm{th,i}}} \sim \frac{1}{\rho_\mathrm{i}} \left| \frac{v_\perp}{\Omega_{\mathrm{c,i}}} - \frac{v'_\perp}{\Omega_{\mathrm{c,i}}} \right| \sim \frac{1}{k_\perp \rho_\mathrm{i}}$$

[Schekochihin et al., PPCF (2008)]

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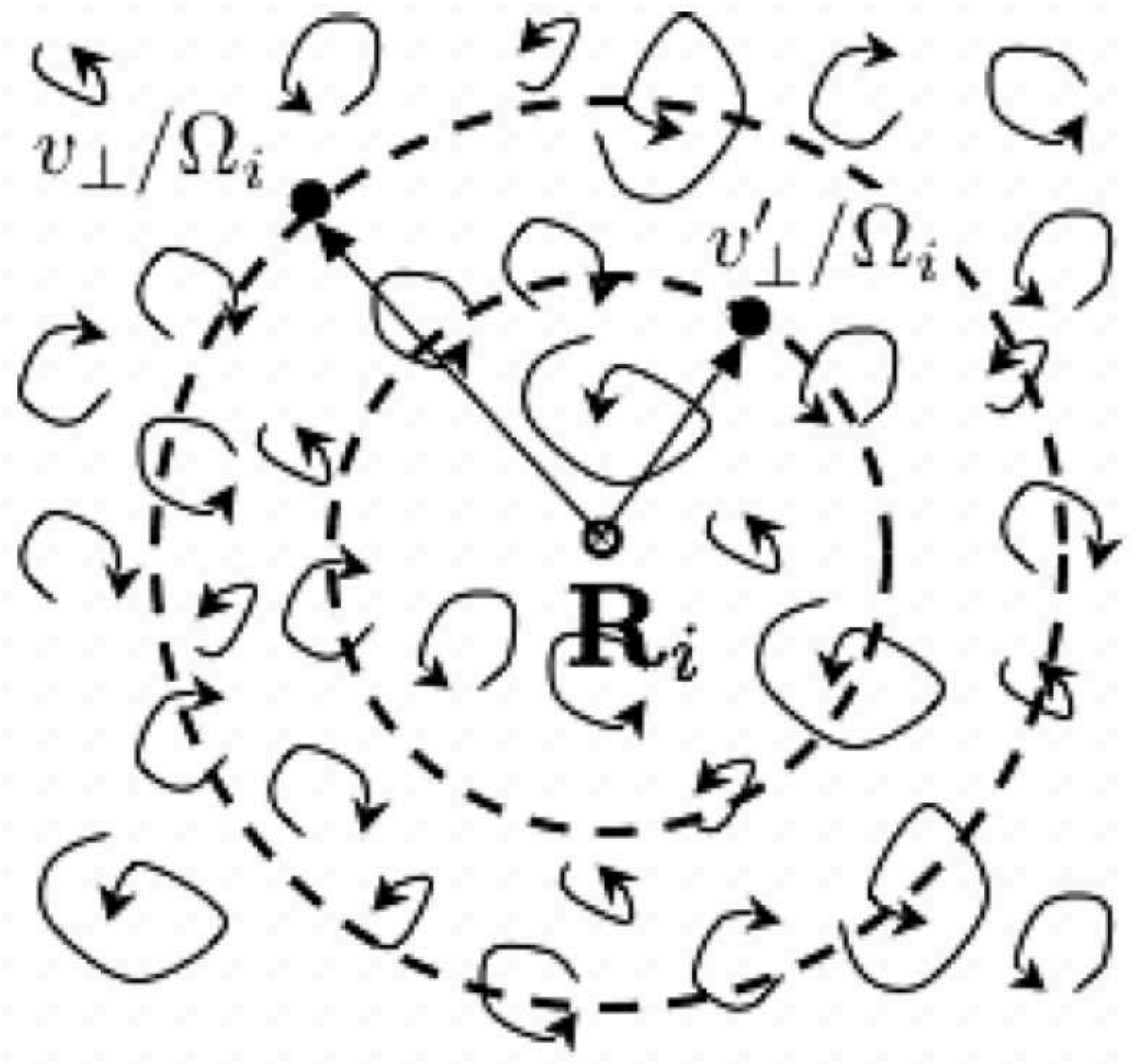

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[Schekochihin et al., PPCF (2008)]

↓

the phase-space cascade induced by nonlinear phase mixing will obey the following relation in Fourier-Hermite space [Cerri et al., ApJL (2018)]

$$m_\perp \propto k_\perp^2$$

⚠ *this does not need any particular ordering nor depends on the scaling of background fluctuations!*

The hybrid-kinetic model

- **Fully kinetic ions** (Vlasov equation for $f_i(\boldsymbol{x},\boldsymbol{v},t)$ — *Eulerian approach: 3D-3V phase-space grid*)

$$\frac{\partial f_\mathrm{i}}{\partial t} + \boldsymbol{v}\cdot\boldsymbol{\nabla} f_\mathrm{i} + \frac{e}{m_\mathrm{i}}\left(\boldsymbol{E} + \frac{\boldsymbol{v}}{c}\times\boldsymbol{B}\right)\cdot\frac{\partial f_\mathrm{i}}{\partial \boldsymbol{v}} = 0$$

- **Electron fluid** (generalized Ohm's law for $\boldsymbol{E}$ — *with finite electron-intertia effects*)

$$\left(1 - d_\mathrm{e}^2\nabla^2\right)\boldsymbol{E} = -\frac{\boldsymbol{u}_\mathrm{i}\times\boldsymbol{B}}{c} + \frac{\boldsymbol{J}\times\boldsymbol{B}}{enc} - \frac{\boldsymbol{\nabla} p_\mathrm{e}}{en} + \frac{4\pi d_\mathrm{e}^2}{c^2}\boldsymbol{\nabla}\cdot\left(\boldsymbol{u}_\mathrm{i}\boldsymbol{J} + \boldsymbol{J}\boldsymbol{u}_\mathrm{i} - \frac{\boldsymbol{J}\boldsymbol{J}}{en}\right)$$

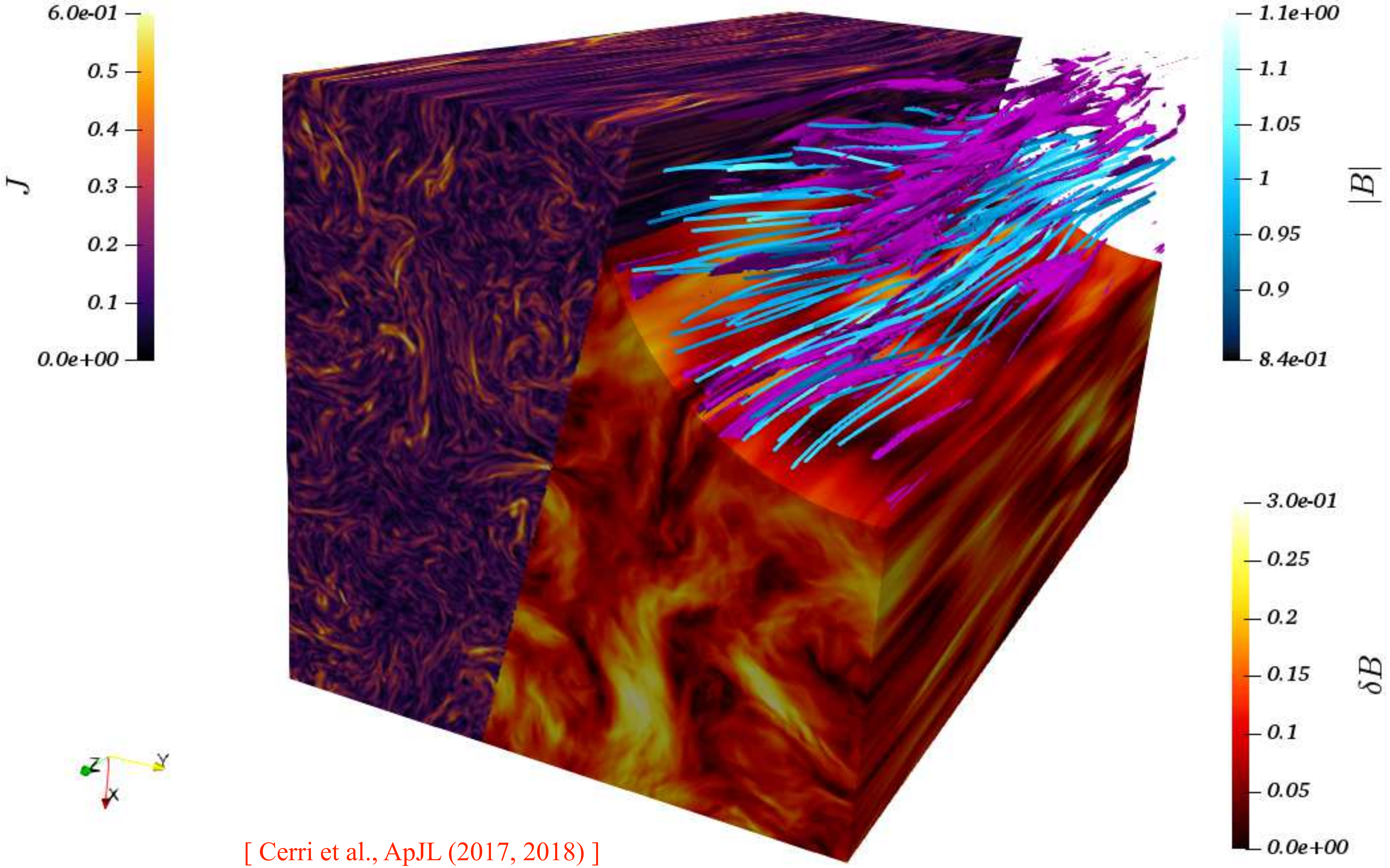
- **Maxwell's equations** (Faraday equation for $\boldsymbol{B}$ + Ampere's law — *w/o displacement current*)

$$\frac{\partial \boldsymbol{B}}{\partial t} = -c\,\boldsymbol{\nabla}\times\boldsymbol{E} \qquad\qquad \boldsymbol{\nabla}\times\boldsymbol{B} = \frac{4\pi}{c}\boldsymbol{J}$$

- Quasi-neutrality is assumed ($n_e = n_i \doteq n$) [$n(\boldsymbol{x},t)$ and $\boldsymbol{u}_\mathrm{i}(\boldsymbol{x},t)$ are $\boldsymbol{v}$-space moments of $f_\mathrm{i}(\boldsymbol{x},\boldsymbol{v},t)$]
- Isothermal closure for the electron fluid ($p_e = n\,T_{e,0}$) [$T_{e,0} = T_{i,0}$ at $t = 0$]
- Reduced mass ratio ($m_i\,/\,m_e = 100$); [$d_e^2 = m_e\,c^2\,/\,4\pi n_e e^2$ = electron inertial length]
- Small-scale dissipation: numerical filters [spectral-like filters (Lele, JCP 1992)]

☞ for more details: Califano & Cerri, Chapter 5 in "Space and Astrophysical Plasma Simulation. Methods, Algorithms, and Applications", Springer (2023)

3D-3V hybrid-Vlasov-Maxwell turbulence simulations



[Cerri et al., ApJL (2017, 2018)]

Cascade of δf in 6D phase space

[Cerri, Kunz & Califano, ApJL (2018)]

$$\delta f(\boldsymbol{x},\boldsymbol{v}) \doteq f(\boldsymbol{x},\boldsymbol{v}) - F_{\rm M}(\boldsymbol{x},\boldsymbol{v})$$

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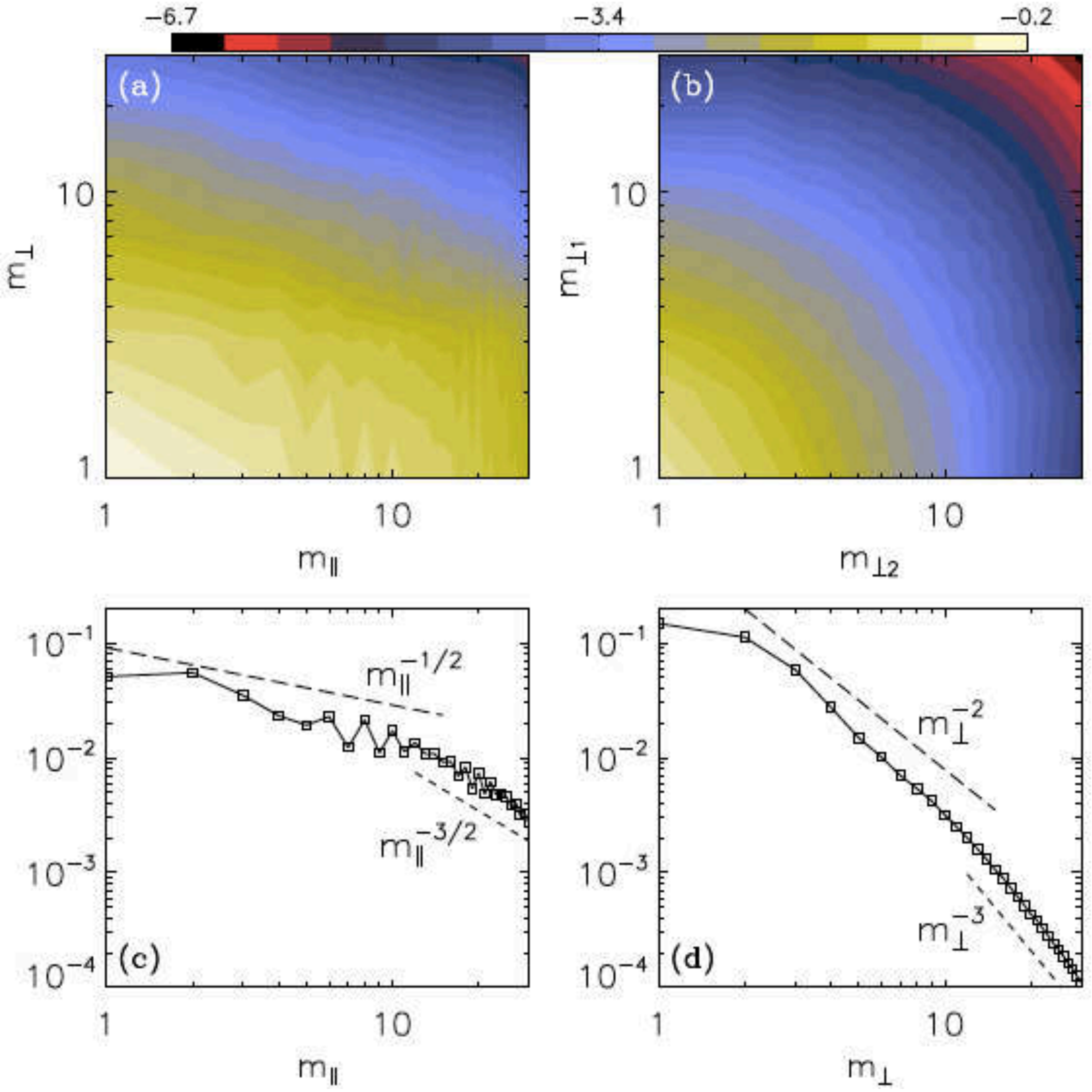

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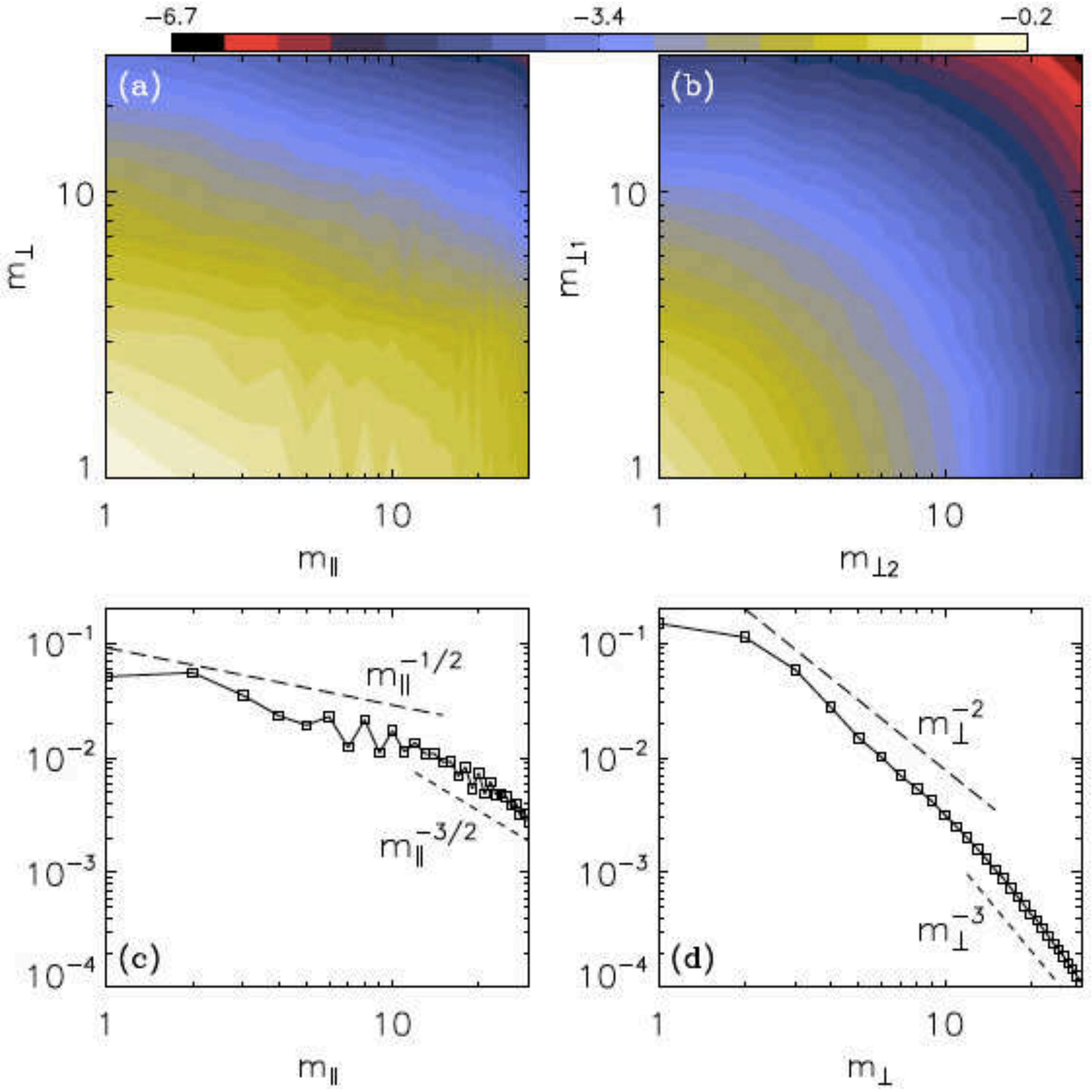
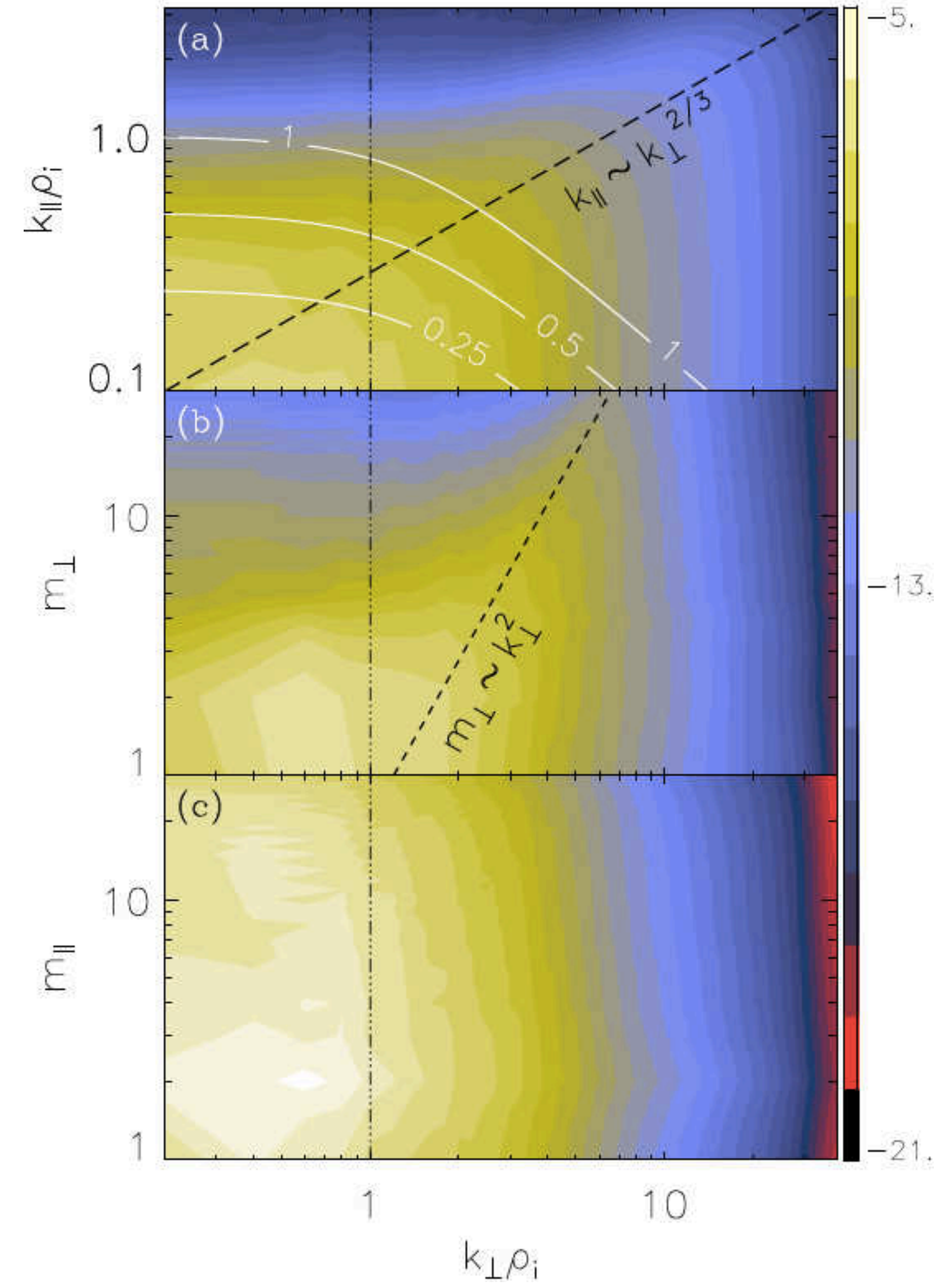
$\log|\delta f_{\boldsymbol{m}}|^2$



$$\delta f(\boldsymbol{x},\boldsymbol{v}) \doteq f(\boldsymbol{x},\boldsymbol{v}) - F_{\mathrm{M}}(\boldsymbol{x},\boldsymbol{v})$$

$\log|\delta f_{\boldsymbol{k},\boldsymbol{m}}|^2$



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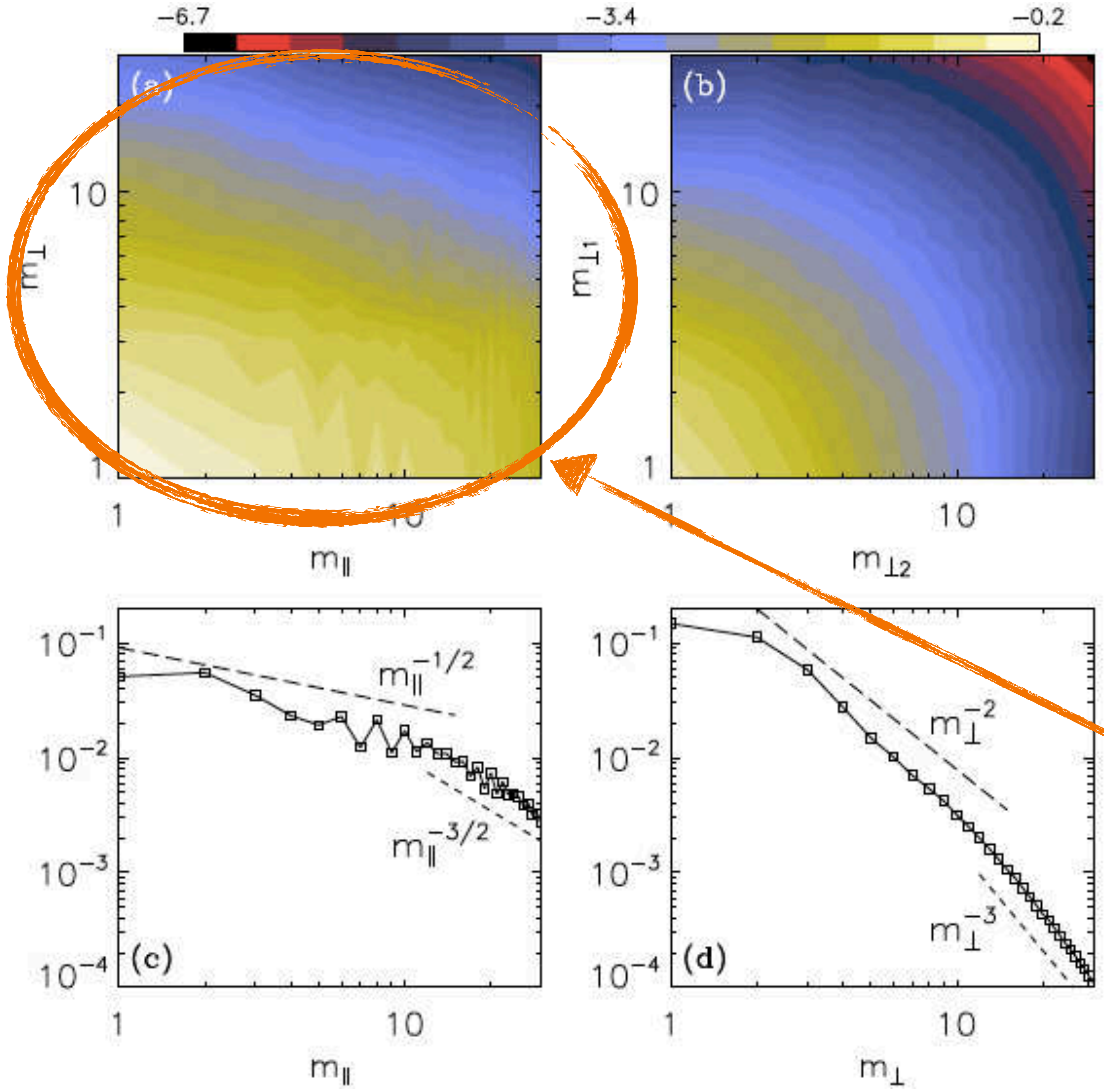

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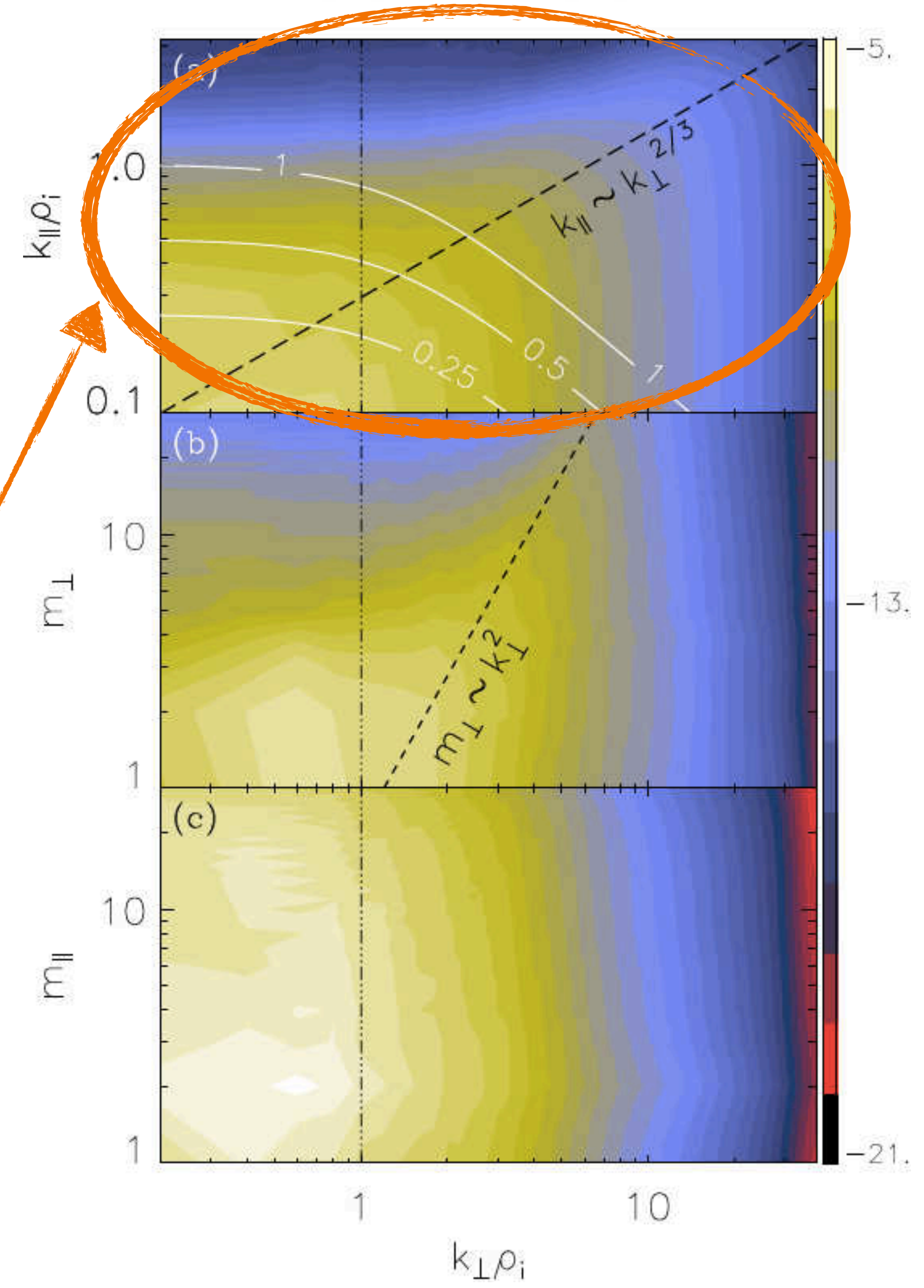


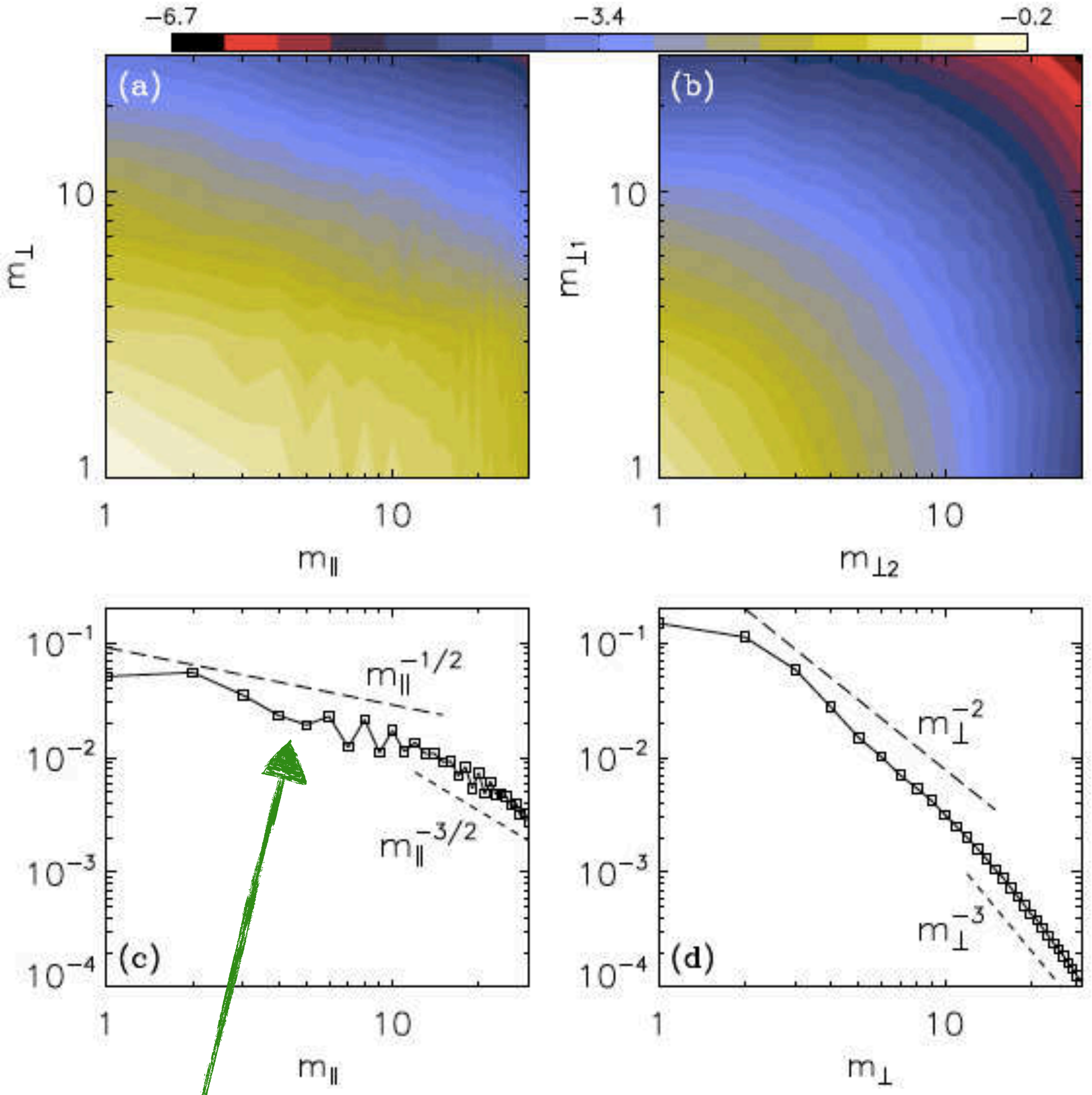
Anisotropic cascade

(w.r.t. $\boldsymbol{B}$) in both $\boldsymbol{m}$ and $\boldsymbol{k}$

Cascade of δf in 6D phase space

[Cerri, Kunz & Califano, ApJL (2018)]

$$\log|\delta f_{\boldsymbol{m}}|^2$$



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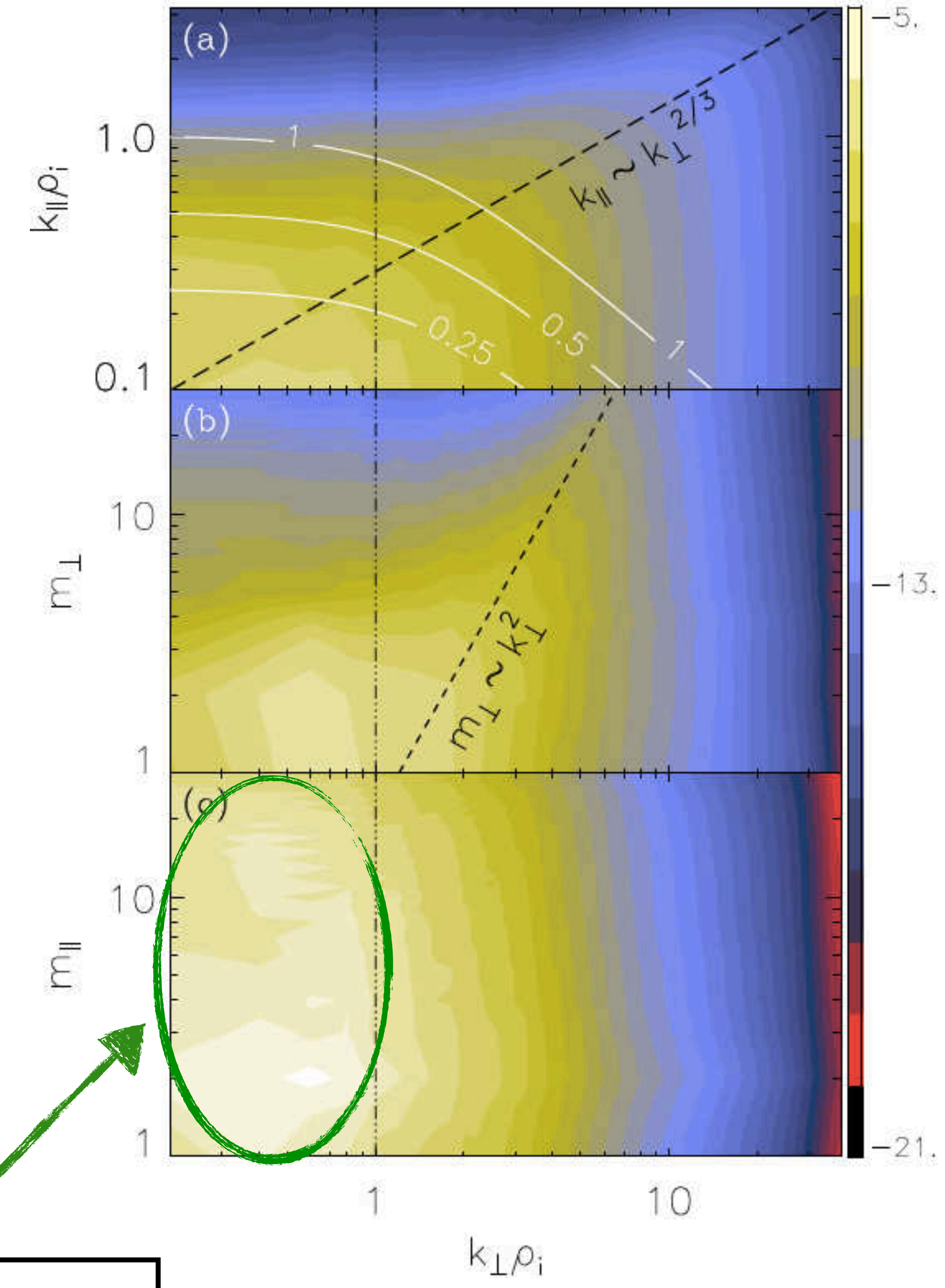


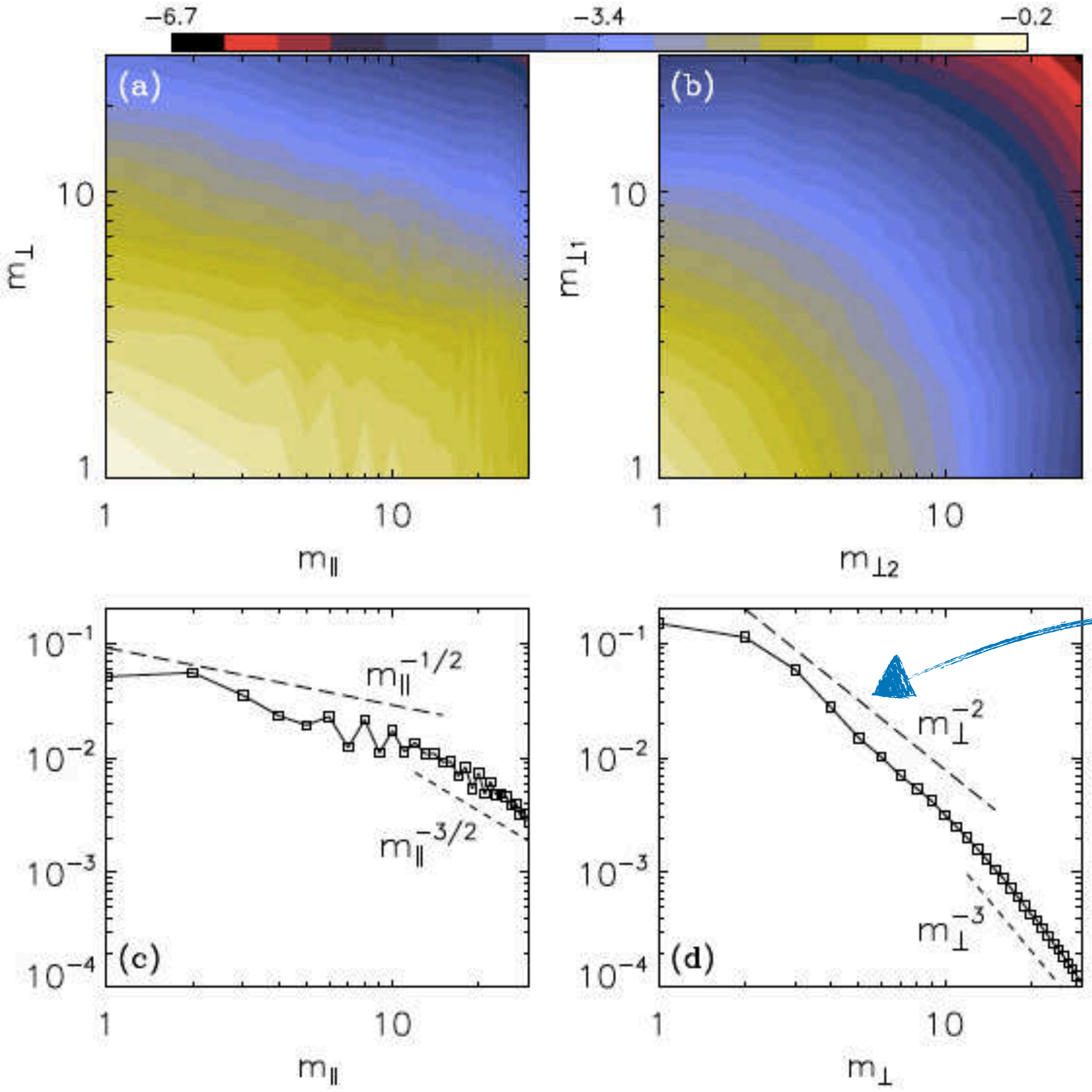
Landau damping

Landau damping
(mainly above ion scales)

Cascade of δf in 6D phase space

[Cerri, Kunz & Califano, ApJL (2018)]

$\log |\delta f_{\boldsymbol{m}}|^2$



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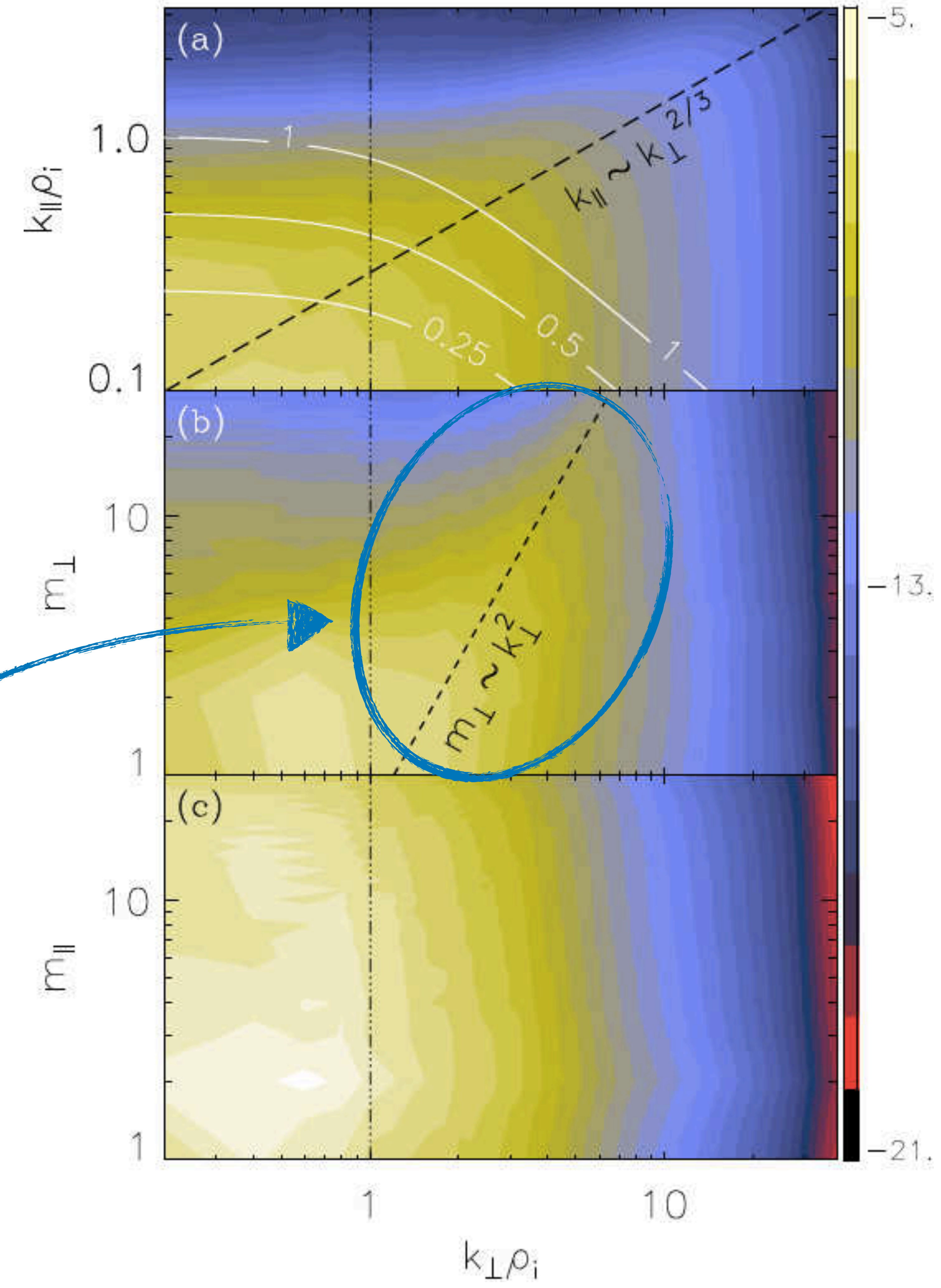


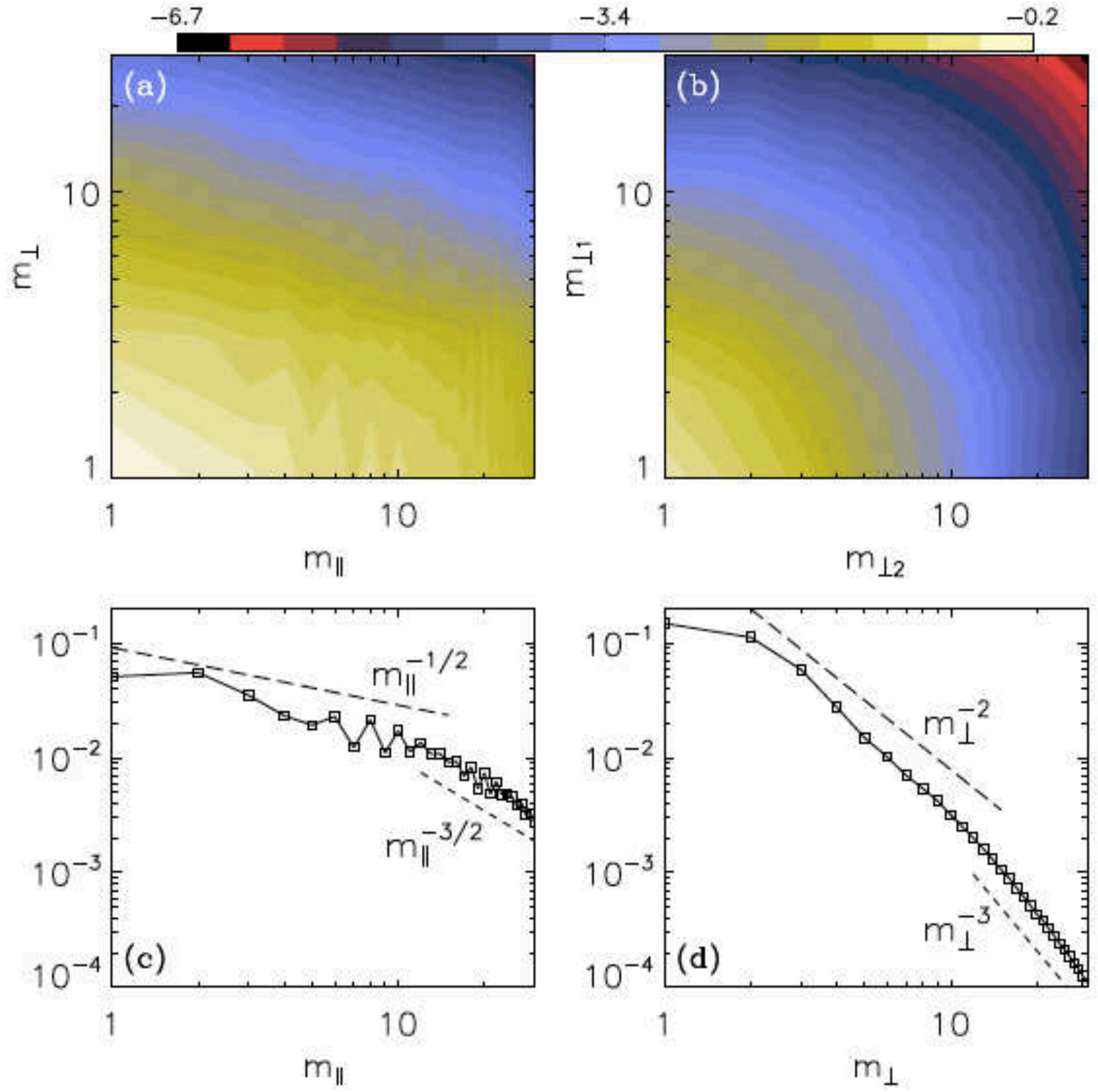
non-linear phase mixing

(at sub-ion scales)

Cascade of δf in 6D phase space

[Cerri, Kunz & Califano, ApJL (2018)]

$\log|\delta f_{\boldsymbol{m}}|^2$



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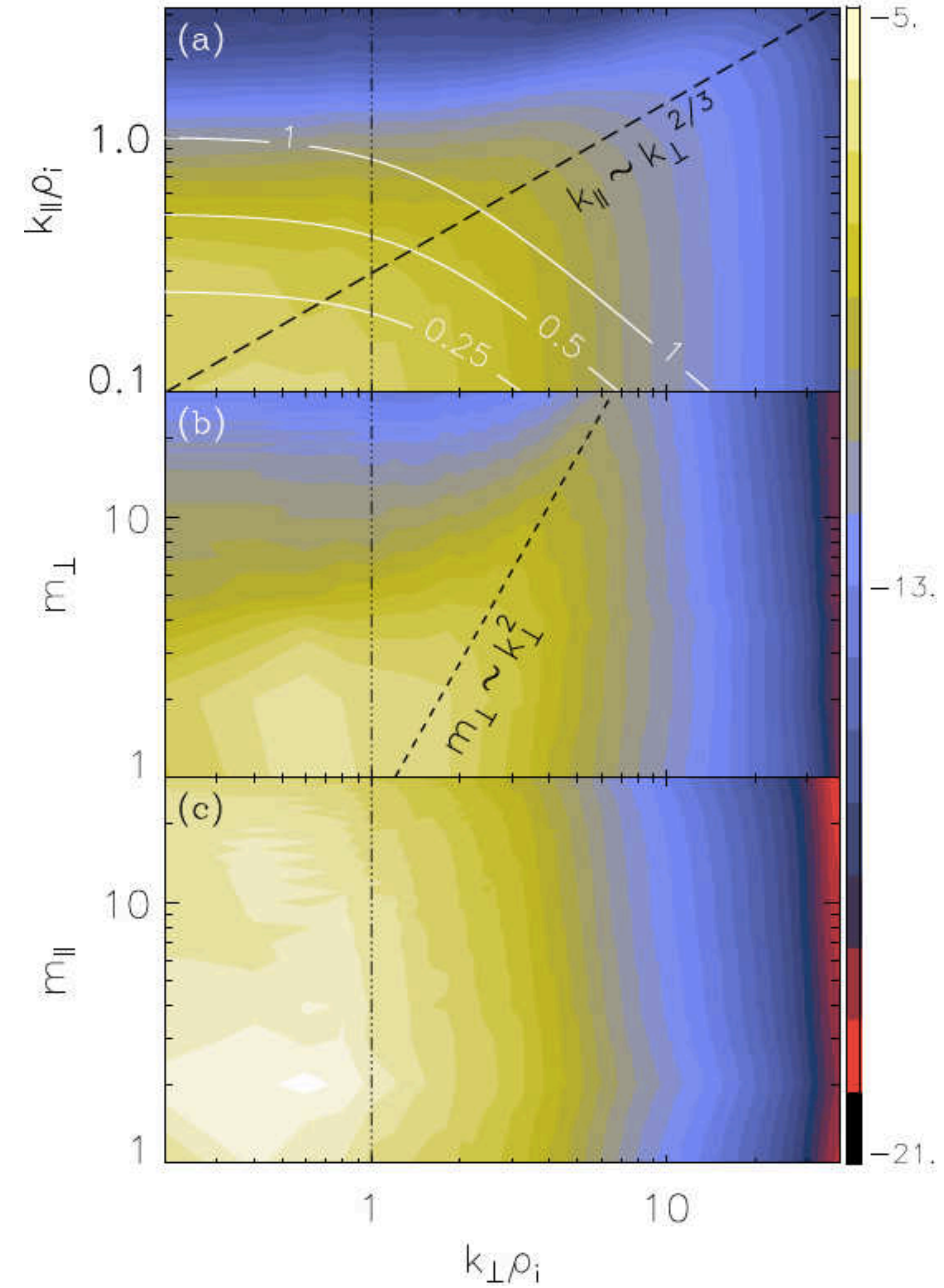


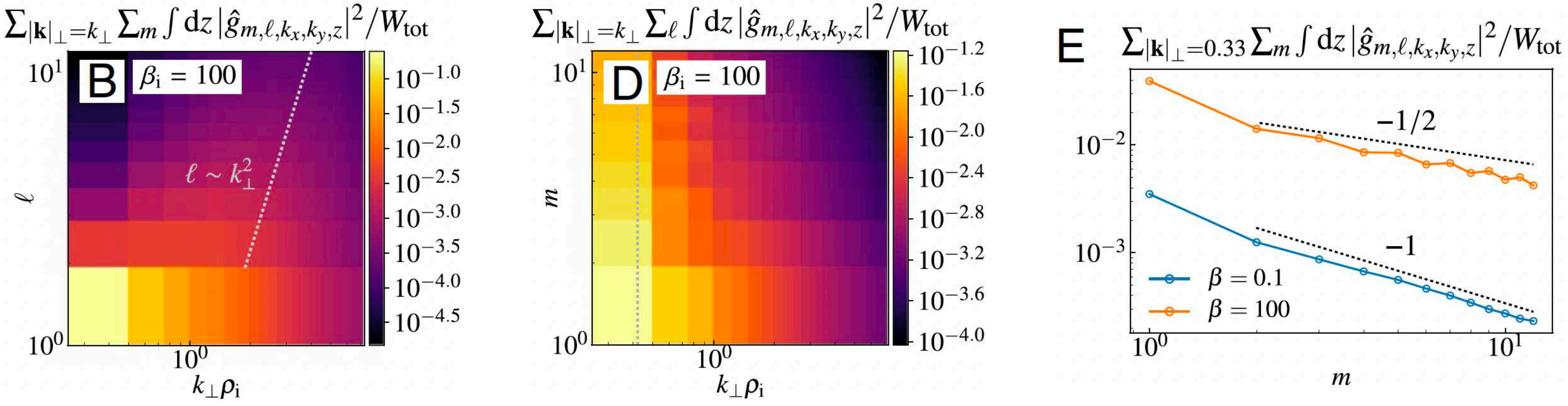
☞ *real- and velocity space are tightly entwined in cascading δf → "phase-space cascade"*

☞ *phase-space cascade of δf induced by turbulent e.m. fluctuations is "6D-anisotropic"*

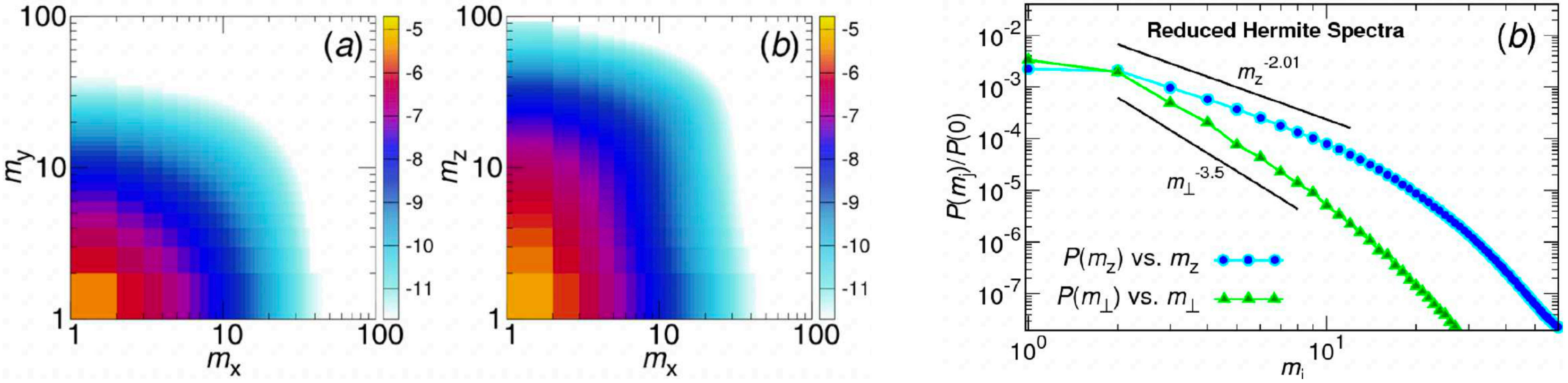
Cascade of δf in *"reduced-dimensionality"* phase space

5D phase space

- **3D e.m. hybrid-gyrokinetics:**
  (3D-2V)
  [Kawazura et al., PNAS (2019)]



- **2D e.m. hybrid-kinetics:**
  (2D-3V)
  [Pezzi et al., PoP (2018)]



4D phase space:

2D-2V: electrostatic gyrokinetics (GK) [Tatsuno et al., PRL (2009), PoP (2012) ; Banon-Navarro et al., PRL (2011); Cerri et al., PoP (2014b)]

3D-1V: reduced-GK [Hatch et al., JPP (2014); Loureiro et al., JCP (2016); Meyrand et al., PNAS (2018); Gobrunov & Teaca, JCP (2023); Zhou et al., PNAS (2023)]

3D phase space: **2D-1V** (e.s. reduced-GK) [Watanabe & Sugama, PoP (2004)]

2D phase space: **1D-1V** (Vlasov-Poisson) [Parker & Dellar, JPP (2015); Celebre et al., PoP (2023)]

Final remarks & discussion

☞ *Nonlinearities* [turbulence] *in kinetic systems intrinsically entwine* [the cascade of] *the fluctuating fields* [$\delta E(x)$ and $\delta B(x)$] *in real space with* [a cascade of] *fluctuations of the distrubtion function* [$\delta f(x,v)$] *in the whole* [six-dimensional] *phase space.*

➥ *we call this a "**phase-space cascade**"*

☞ *Such phase-space cascade is likely anisotropic as a result of a combination of various processes simultaneously at play (with different timescales that also depend on the scale); and a reduced dimensionality of the phase space may affect these properties.*

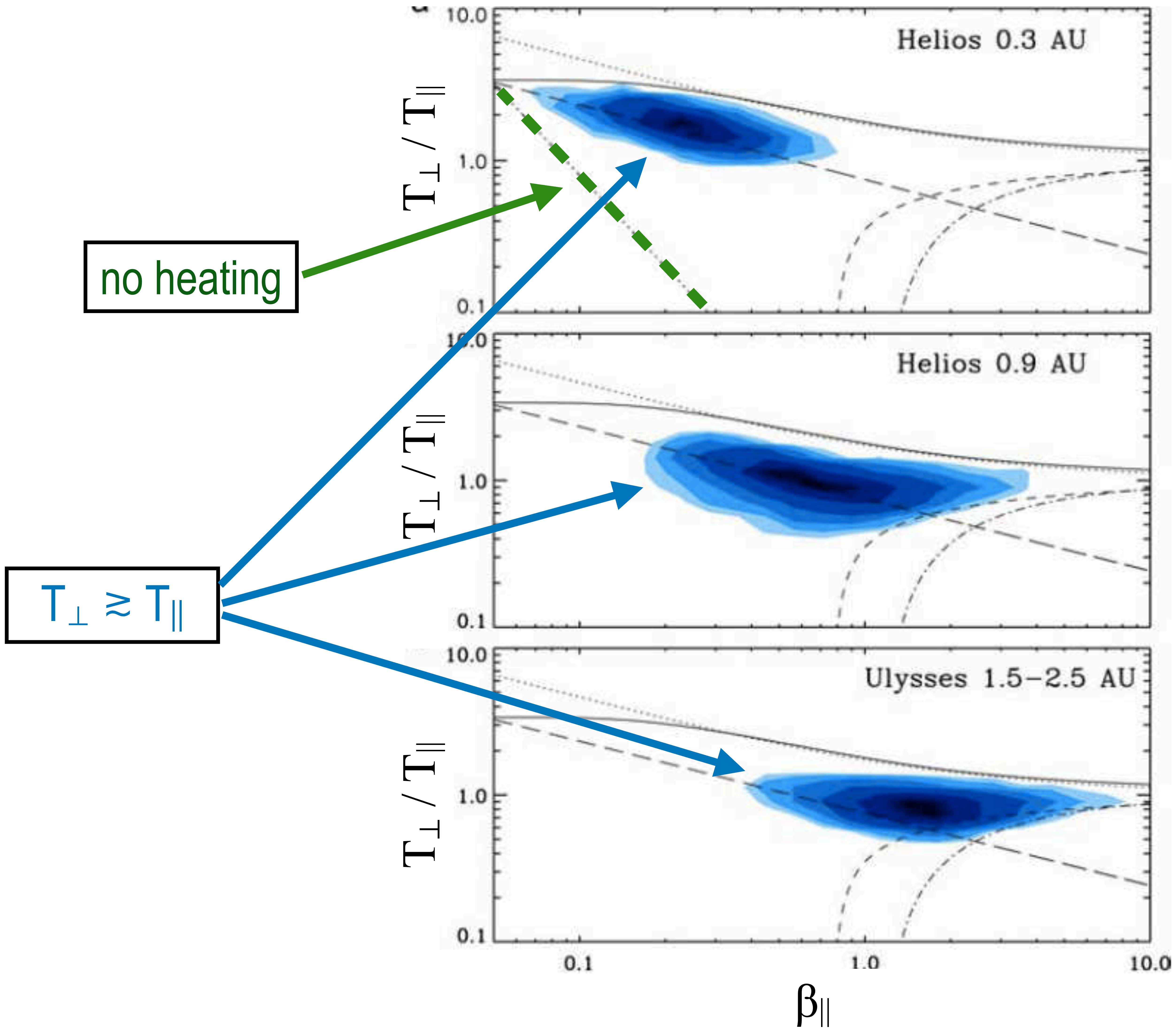
➥ *but still worth looking at one process at a time and in reduced dimensionality*

Phase space is a common (play)ground for our two research communities…
…let's meet there and have fun with nonlinearities!

BACKUP SLIDES

Turbulent heating in the solar wind

[Matteini et al., GRL (2007)]

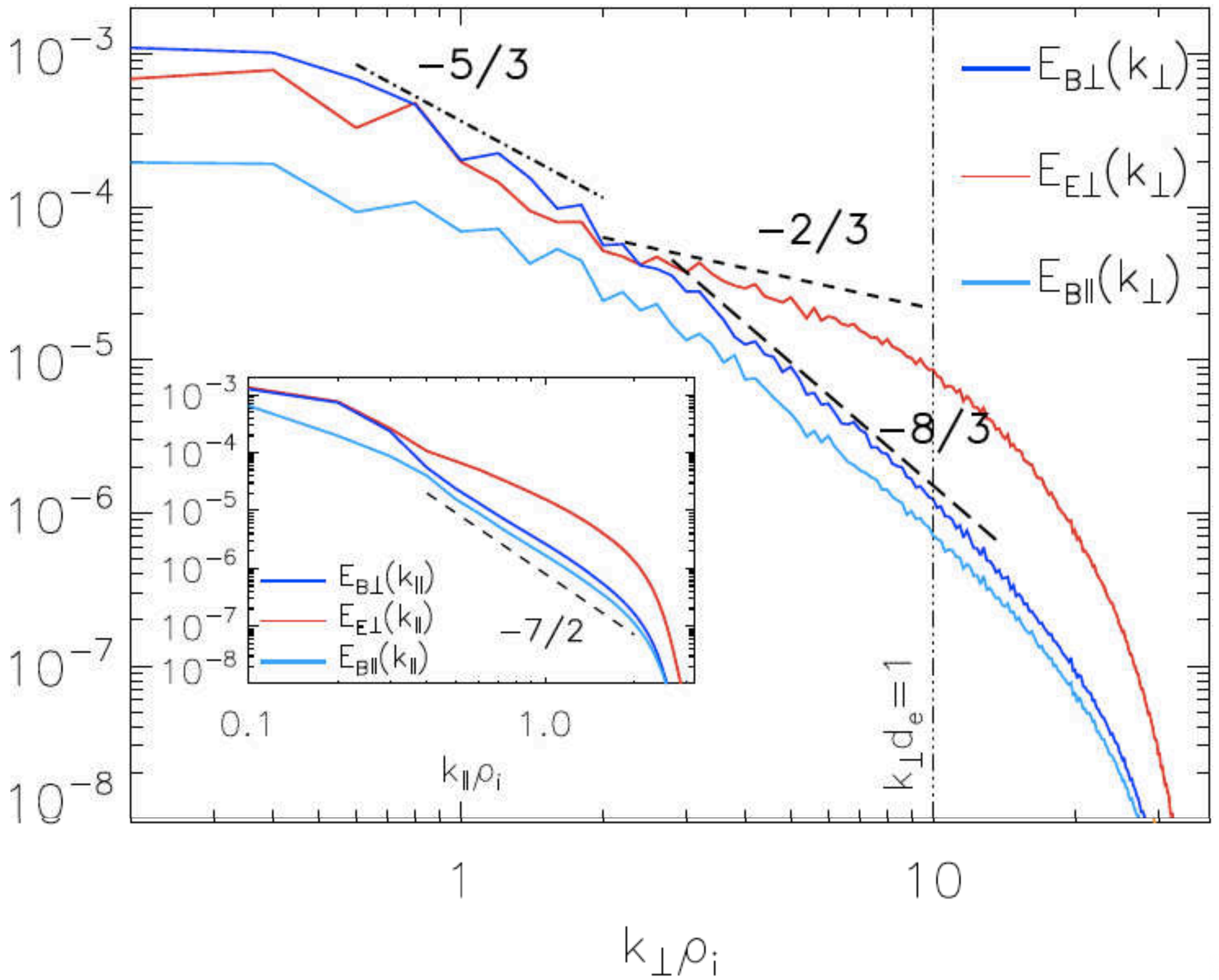


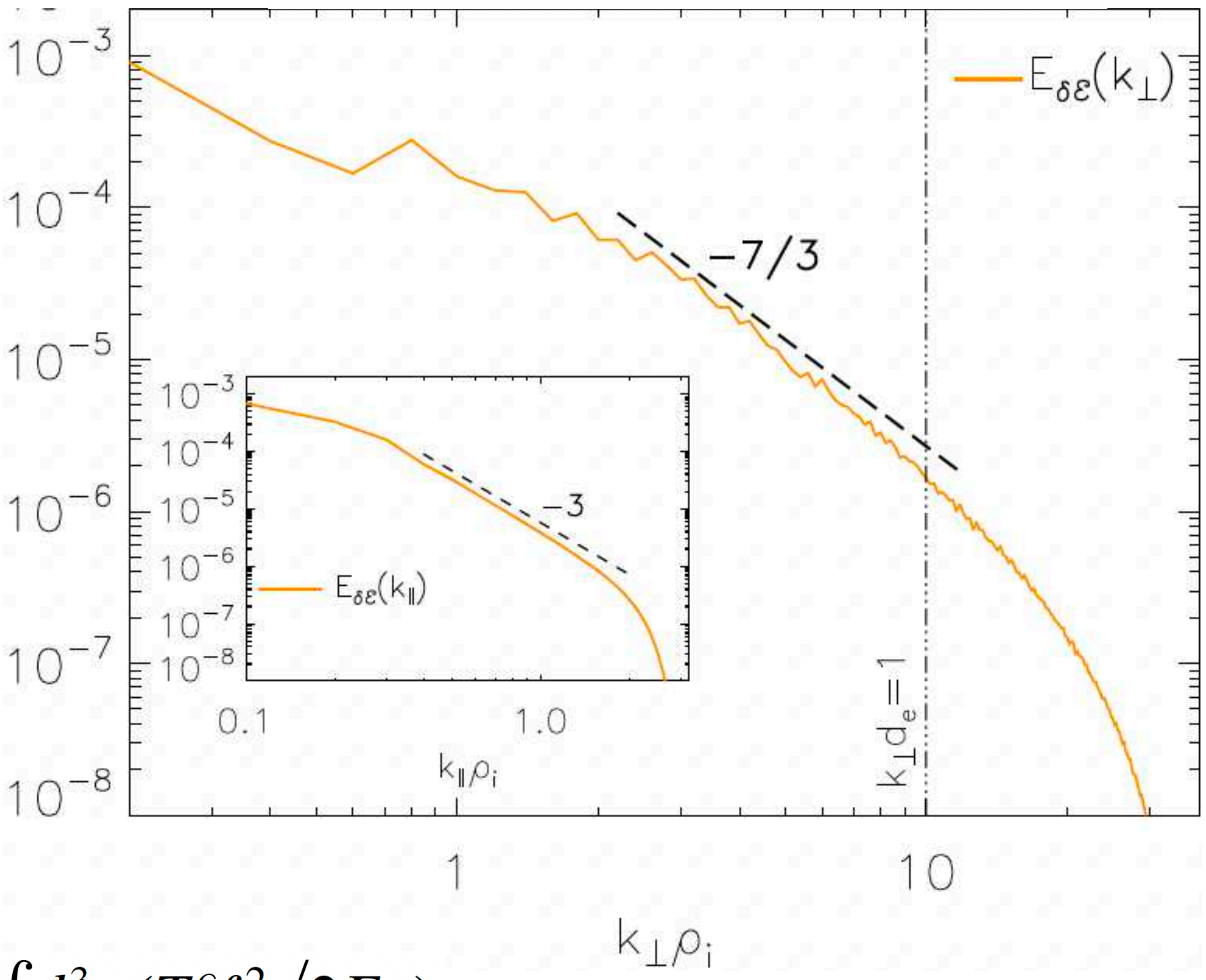
- preferential heating of ions perpendicular to B

Fluctuations′ spectra in the HVM simulation

[Cerri, Kunz & Califano, ApJL (2018)]

$$\delta\mathcal{E} \equiv -\int d^3\mathbf{v}\,(f\ln f - F_M \ln F_M)$$





$$\delta\mathcal{E} \approx \int d^3\mathbf{v}\,(T\delta f^2 / 2F_M)$$

$$\approx \int d^3\mathbf{v}\,(Th^2 / 2F_M) \text{ for } k_\perp\rho_i \gg 1$$

h = non-adiabatic "gyrokinetic" response

Generalized scaling for the "ion-entropy cascade" (1)

[Cerri, Kunz & Califano, ApJL (2018)]

➢ **underlying idea** (Schekochihin et al., PPCF 2008): $\frac{\delta v_\perp}{v_{\rm th,i}} \sim \frac{1}{\rho_{\rm i}}\left|\frac{v_\perp}{\Omega_{\rm c,i}} - \frac{v'_\perp}{\Omega_{\rm c,i}}\right| \sim \frac{1}{k_\perp \rho_{\rm i}}$ ⟷ $m_\perp \propto k_\perp^2$

de-correlation of $v_\perp$-structures
due to de-correlated $k_\perp$-fluctuations

➢ allow ***generalised spectral anisotropy***: $\ell_{\|,\lambda} \propto \lambda^{\alpha/3}$

α = 1 standard KAW anisotropy
α = 2 intermittency-corrected case
α = 3 constant anisotropy

➢ assume ***critically balanced KAW cascade***: $\tau_{{\rm nl},\lambda} \sim \tau^{(\alpha)}_{{\rm KAW},\lambda} \propto \lambda^{1+\alpha/3}$

↑ no particular model odering assumed this far

↓ *gyrokinetic* ordering assumed in the following

➢ ion-entropy non-linear timescale: $\widetilde{\tau}_{h,\lambda} \sim \left(\frac{\rho_{\rm i}}{\lambda}\right)^{1/2} \tau^{(\alpha)}_{{\rm KAW},\lambda}$

weighting of the nonlinear cascade time
due to ring-average in GK nonlinearity

➢ ***ion-entropy cascade time***: $\tau^{(\alpha)}_{h,\lambda} \sim \left(\frac{\rho_{\rm i}}{\lambda}\right) \tau^{(\alpha)}_{{\rm KAW},\lambda} \sim \lambda^{\alpha/3}$

small changes in the non-adiabatic GK response (h_λ)
that accumulate as a random walk
to produce a change of order unity ($\Delta h_\lambda / h_\lambda \sim 1$)

Generalized scaling for the "ion-entropy cascade" (2)

[Cerri, Kunz & Califano, ApJL (2018)]

➢ assume ***constant entropy flux through scales***: $h_\lambda^2/\tau_{h,\lambda} \sim \varepsilon_h = \text{const}$

➢ derive ***scalings for non-adiabatic GK response***: $h_\lambda^{(\alpha)} \propto \lambda^{\alpha/6}$

accounting for "generalised" spectral anisotropy

➢ derive ***phase-space spectra of ion-entropy cascade***:

$$E_h(k_\perp) \propto k_\perp^{-(3+\alpha)/3} \qquad E_h(k_\parallel) \propto k_\parallel^{-2} \qquad E_h(m_\perp) \propto m_\perp^{-(6+\alpha)/6}$$

α = 1 standard KAW anisotropy: $k_\parallel \propto k_\perp^{1/3}$ $\quad E_h \propto k_\perp^{-4/3}$ $\quad E_h(m_\perp) \propto m_\perp^{-7/6}$

α = 2 intermittency-corrected case: $k_\parallel \propto k_\perp^{2/3}$ $\quad E_h \propto k_\perp^{-5/3}$ $\quad E_h(m_\perp) \propto m_\perp^{-4/3}$

α = 3 constant anisotropy: $k_\parallel \propto k_\perp$ $\quad E_h \propto k_\perp^{-2}$ $\quad E_h(m_\perp) \propto m_\perp^{-3/2}$

δf from a single sub-domain of the HVM simulation