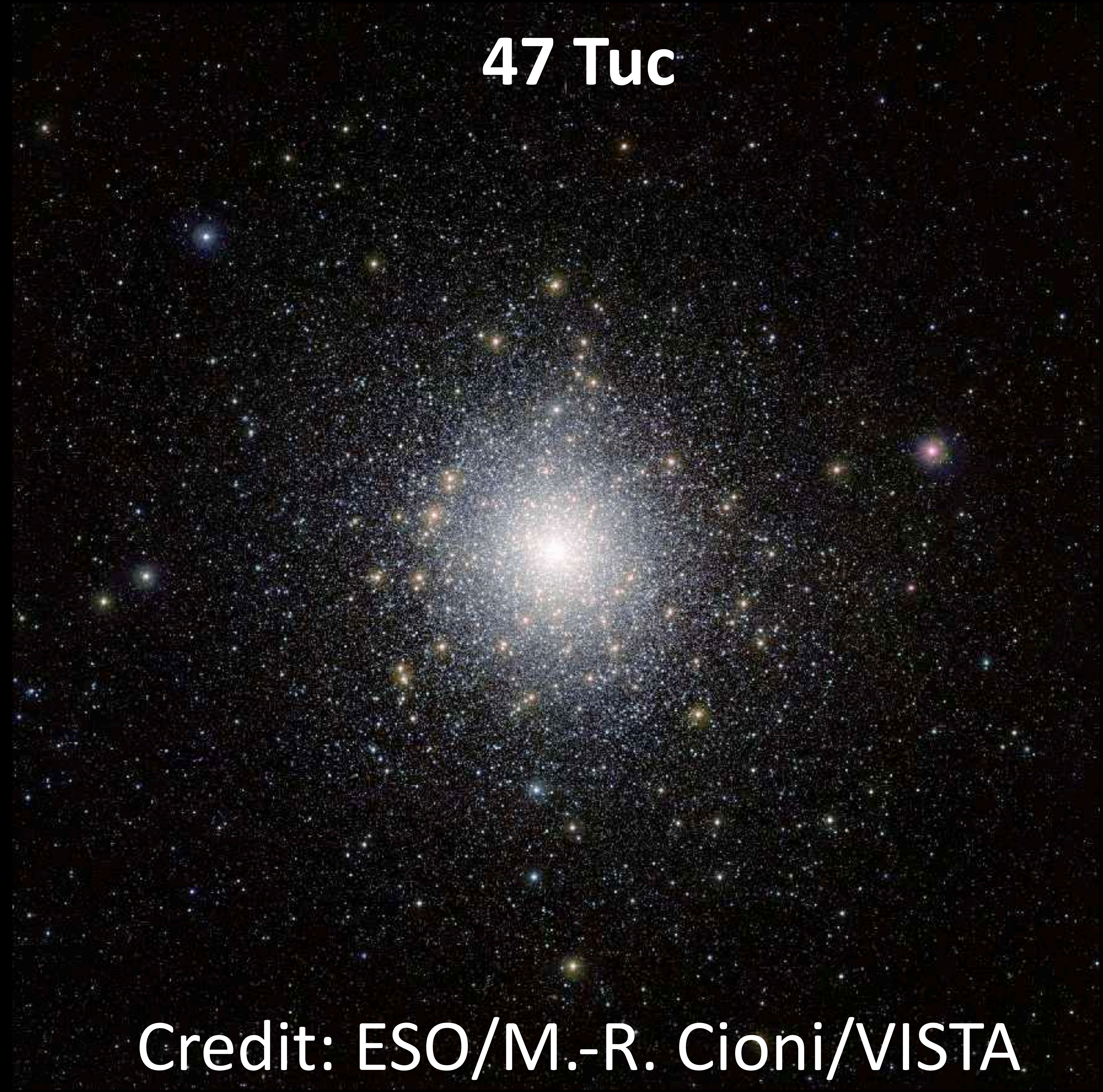
KITP Kinetics 24 Workshop

Monte Carlo Methods for Stellar Dynamics

Carl Rodriguez
University of North Carolina at Chapel Hill

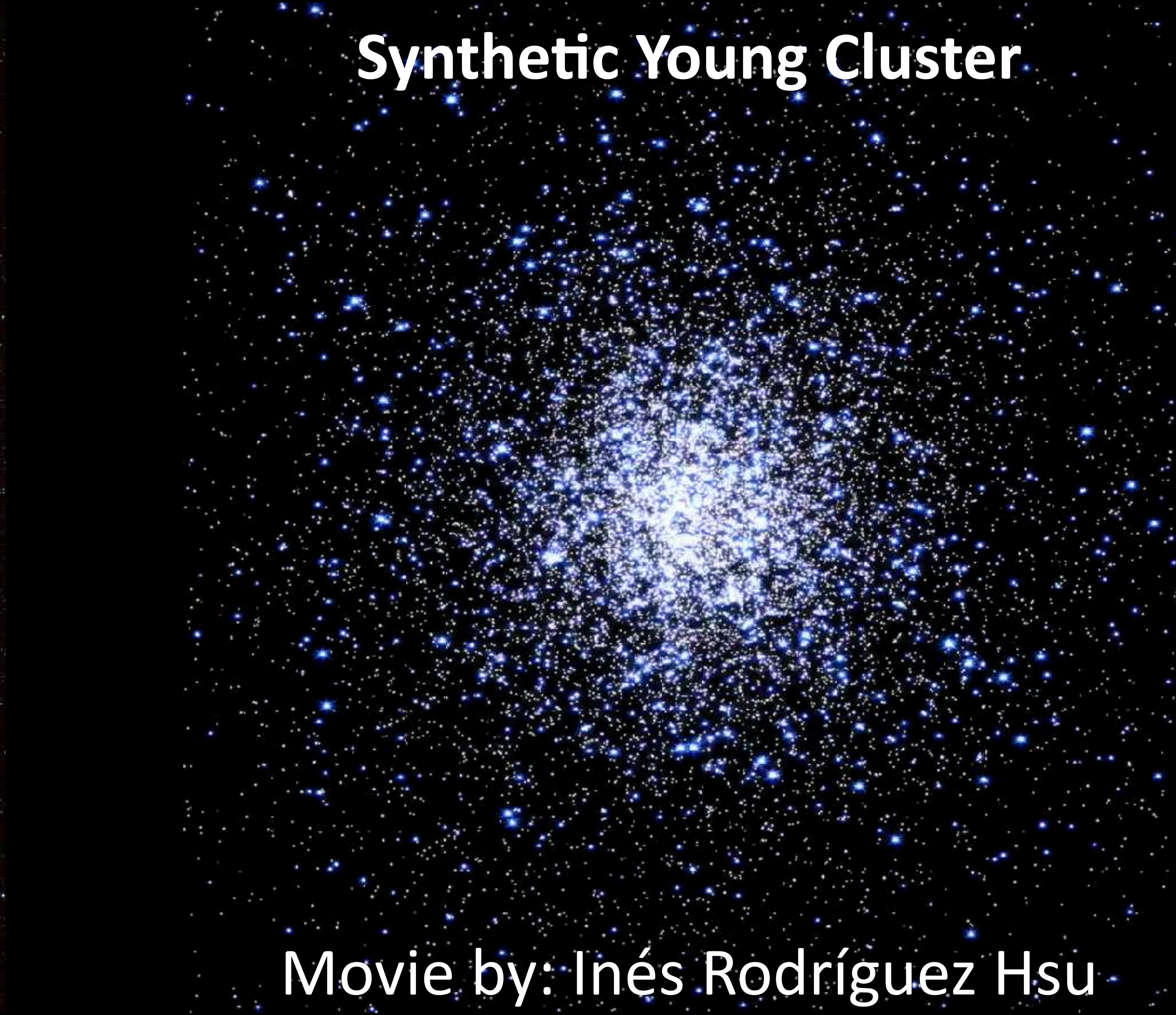
The Problem: Spherical(ish) Star Clusters

47 Tuc



Credit: ESO/M.-R. Cioni/VISTA

Synthetic Young Cluster



Movie by: Inés Rodríguez Hsu

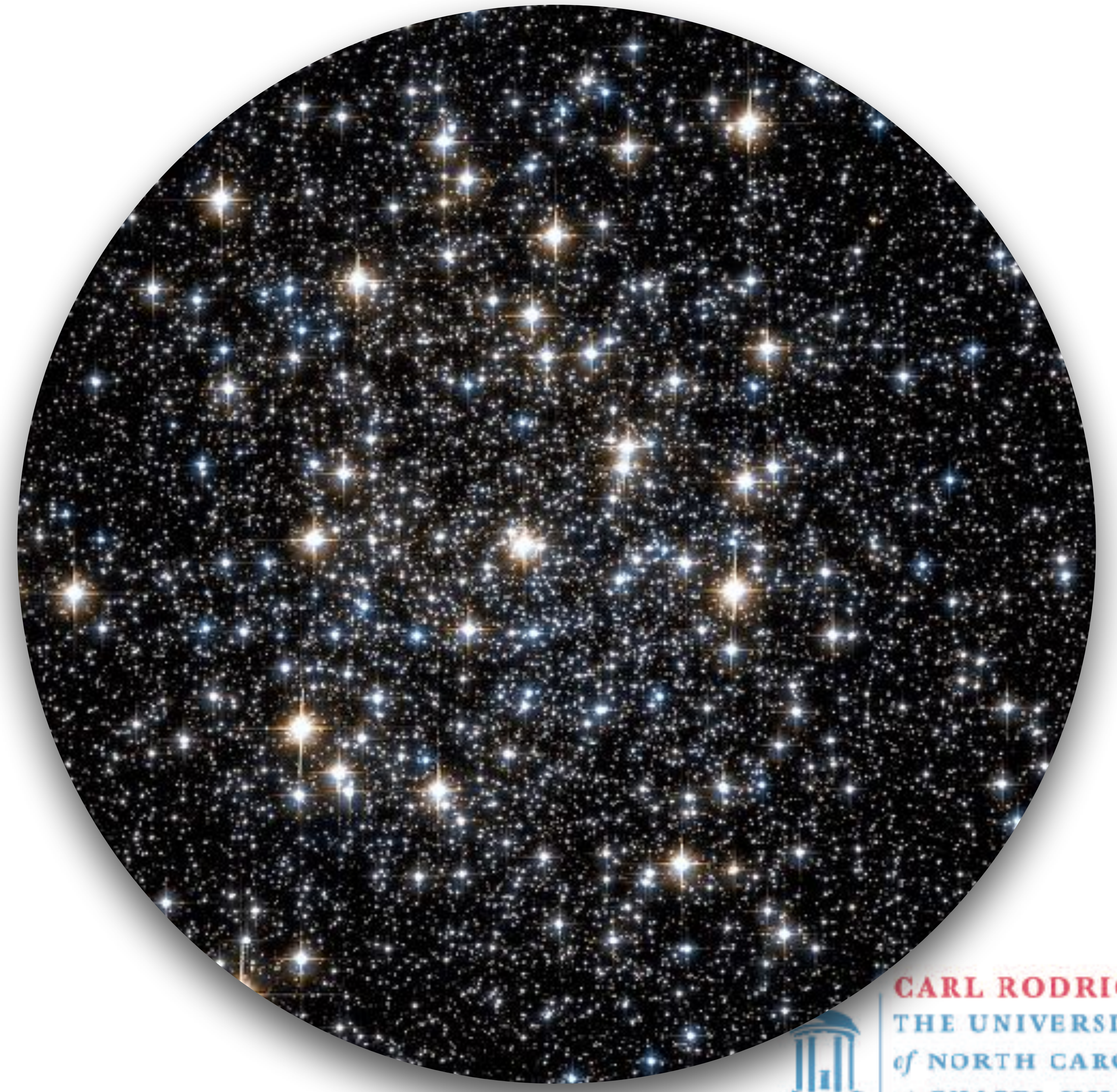
Globular Clusters

- **Old** ($\sim 12\ \mathrm{Gyr}$) & **low metallicity**
- **Massive** ($\sim 10^5$ to $\sim 10^7$ stars and binaries)
- **Compact** ($R_{\mathrm{eff}} \sim \mathrm{few\ pc}$)

M30

47 Tuc

NGC 3201



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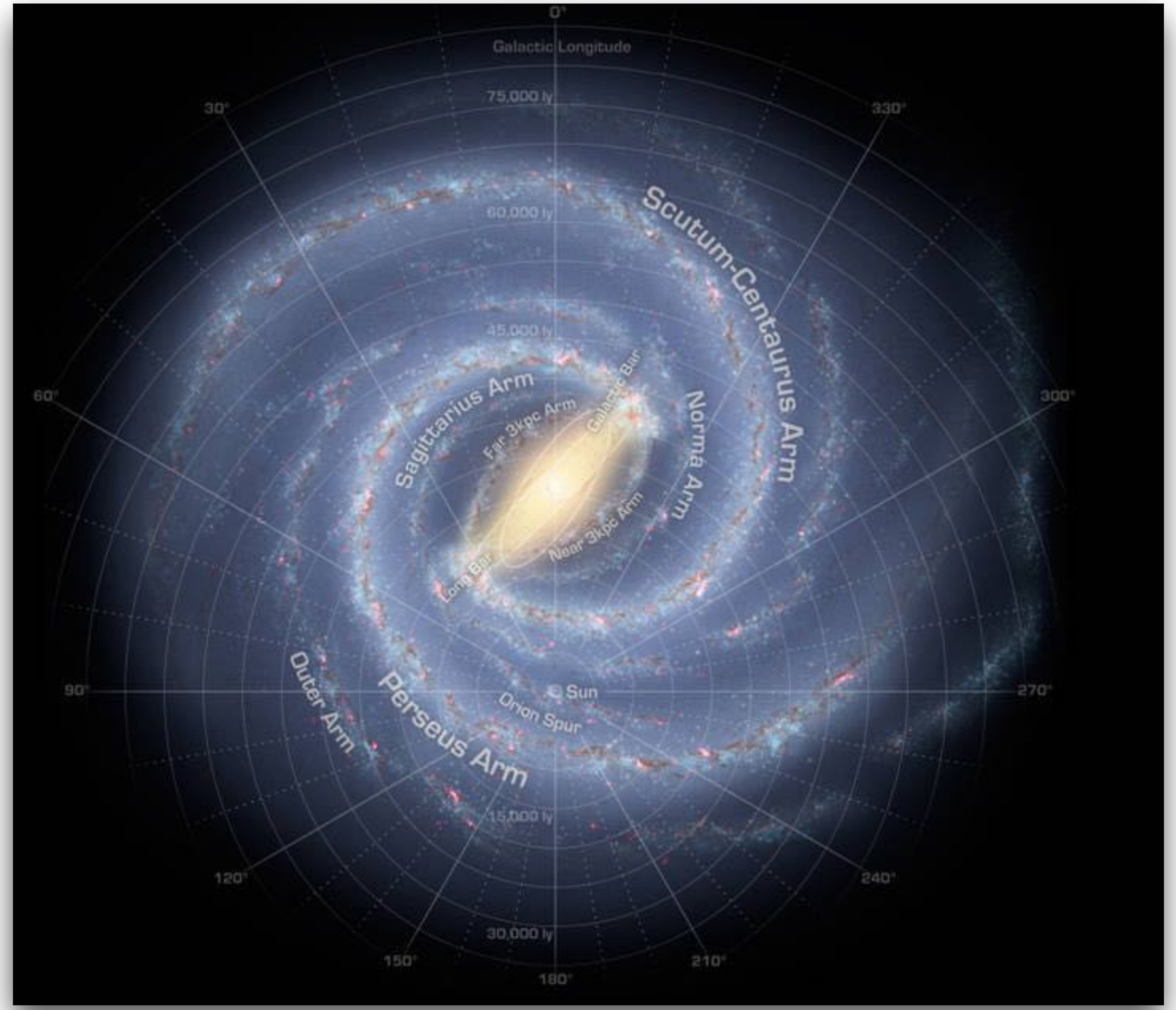
Globular Clusters are everywhere

Found in **almost all** galaxies



Milky Way

NASA/Adler/U. Chicago/Wesleyan/JPL-Caltech

M87

Adam Block/Mt. Lemmon SkyCenter/U. Arizona

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Collisional Systems

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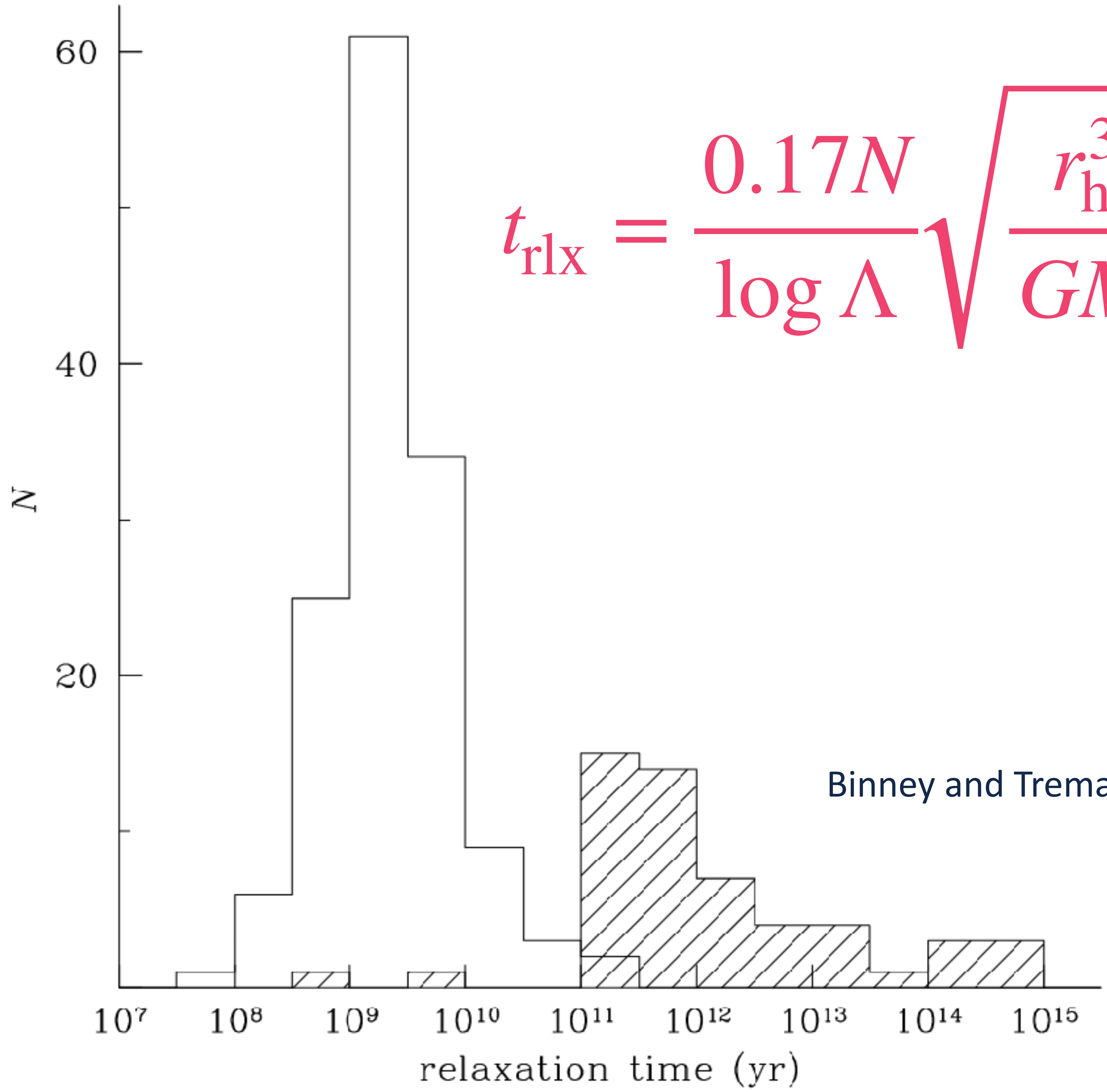
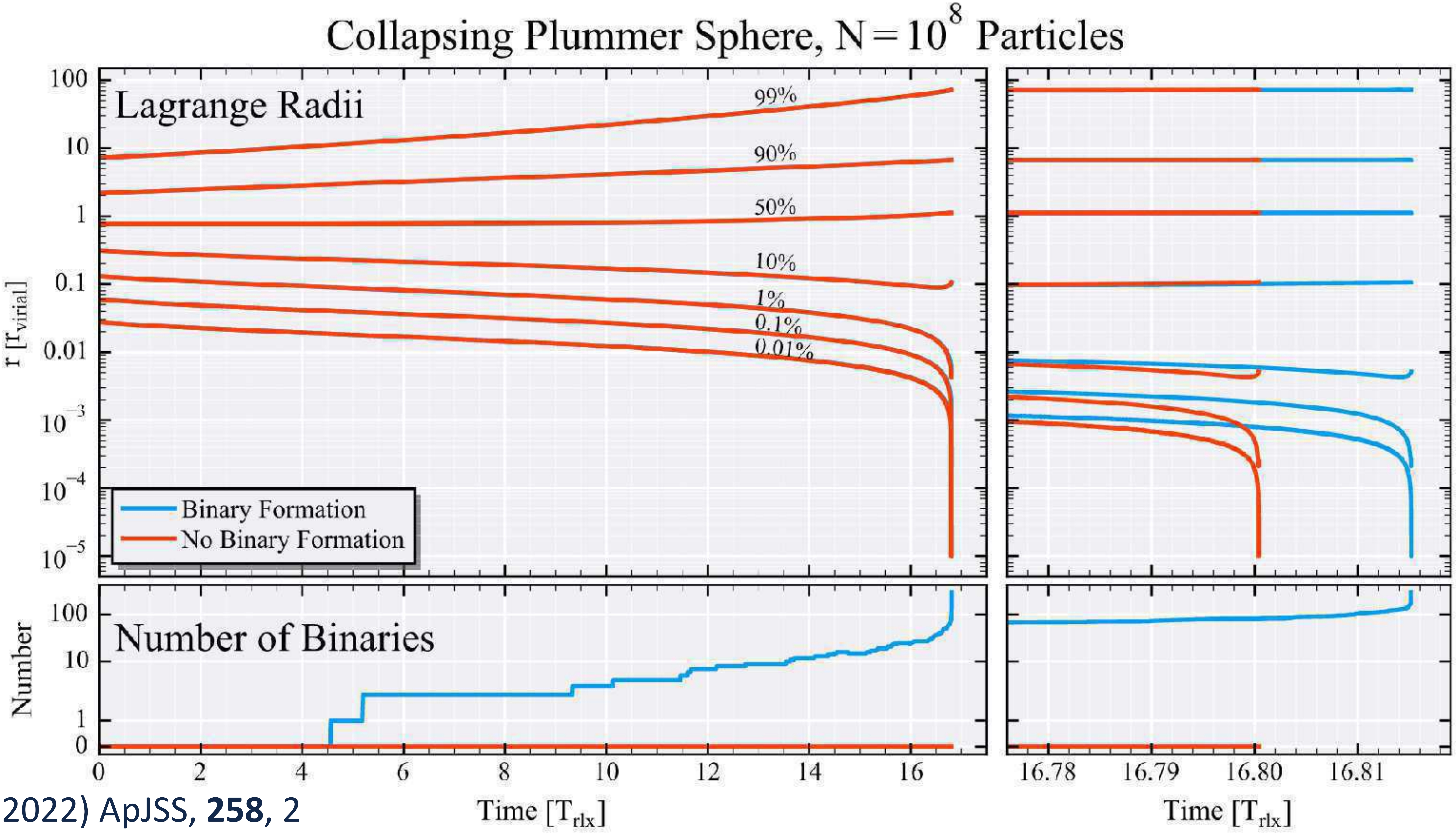
Credit: ESO/M.-R. Cioni/VISTA



$$t_{\rm rlx} = \frac{0.17N}{\log\Lambda}\sqrt{\frac{r_{\rm h}^3}{GM}}$$

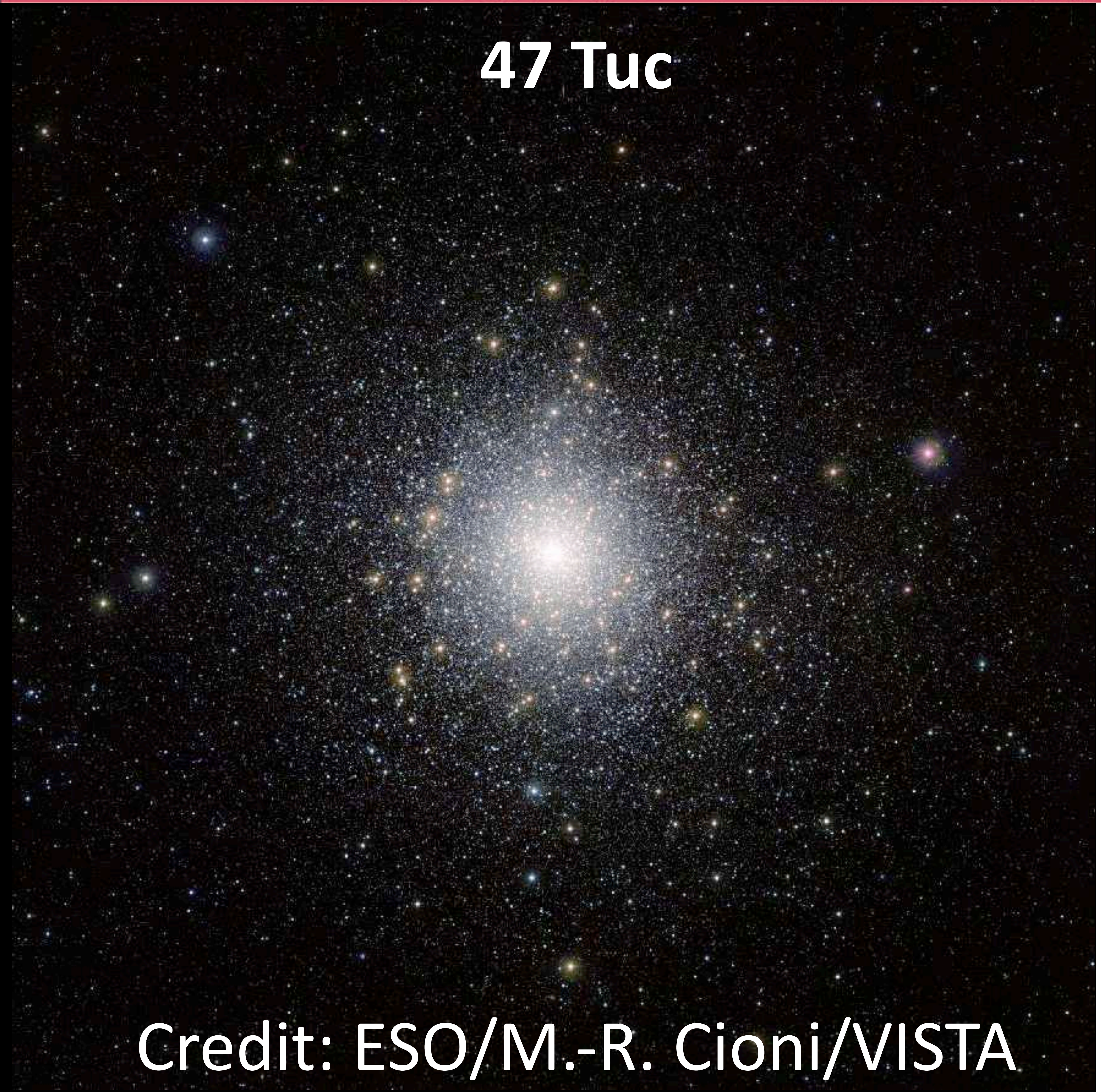
Binney and Tremaine 2008

Collisional Systems



CLR et al. (2022) ApJSS, **258**, 2

The Key Questions



Credit: ESO/M.-R. Cioni/VISTA

Macrophysics Questions

Global Evolution of Star Clusters

- Bulk Properties (radii, masses, etc) globular clusters and galactic nuclei
- Formation and survival
- Destruction (tidal streams)

Microphysics Questions

Production of Unique Stars and Binaries

- Dynamical processes create and modify binary stars
- Physical Collisions

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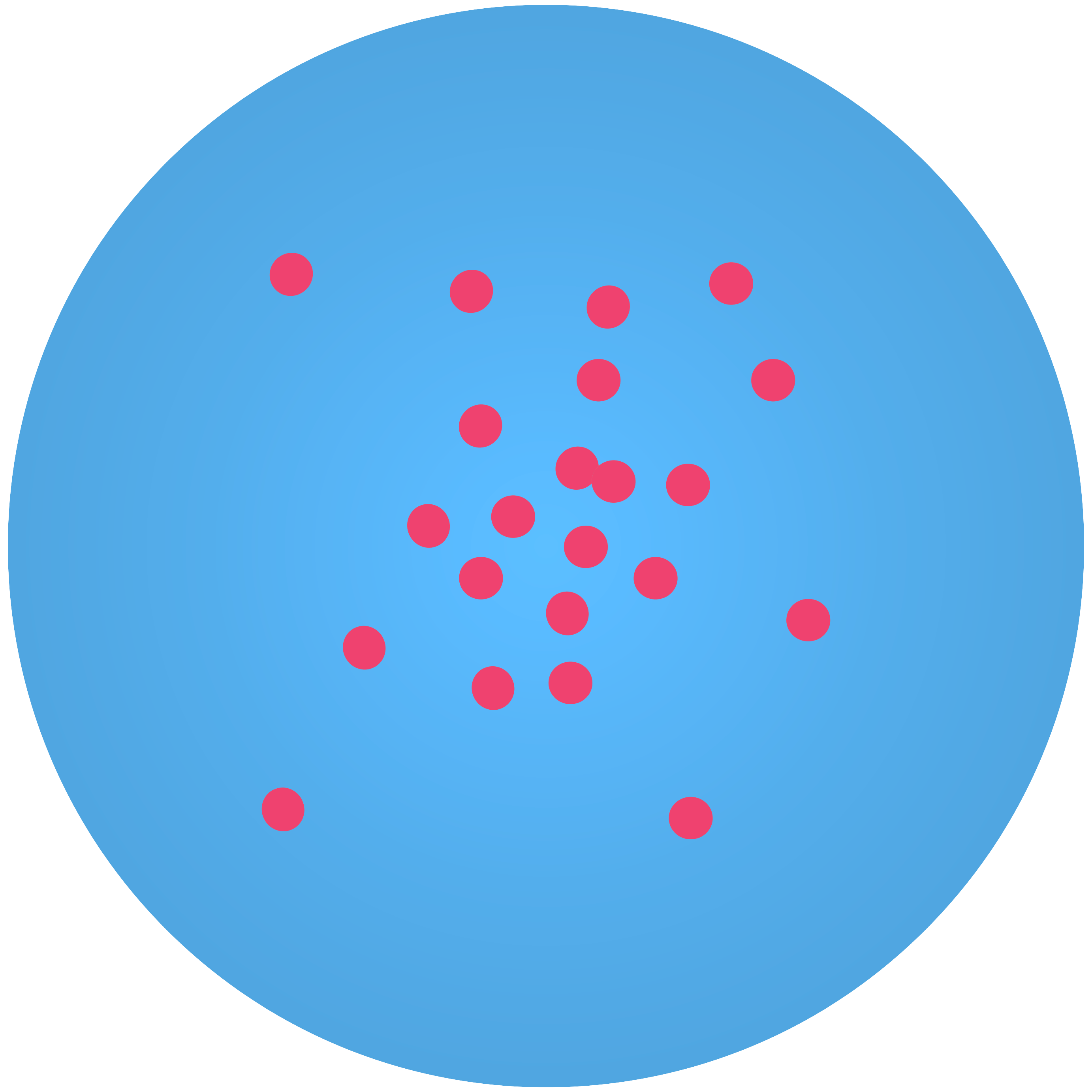
Dynamical Friction



~ 10 Myr

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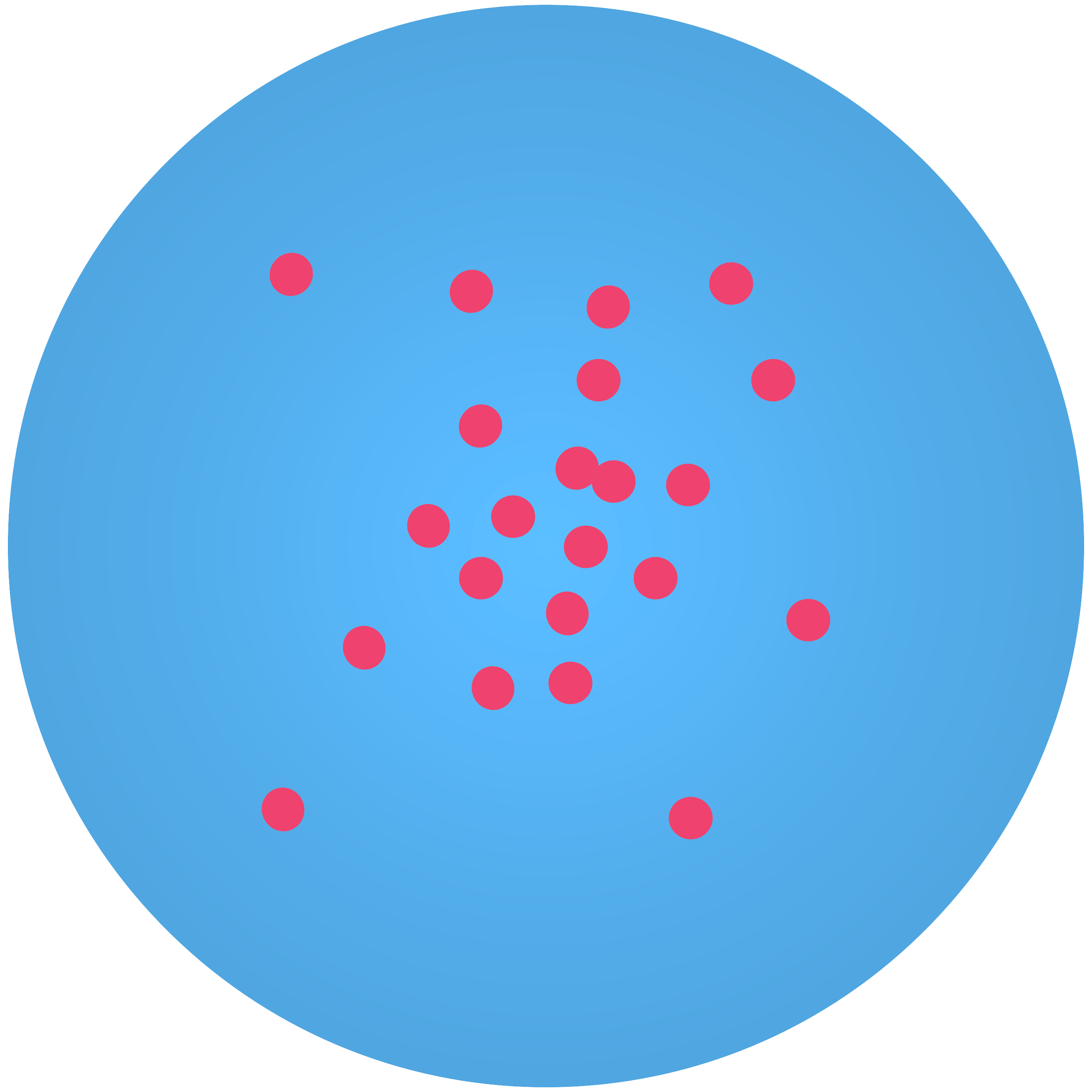

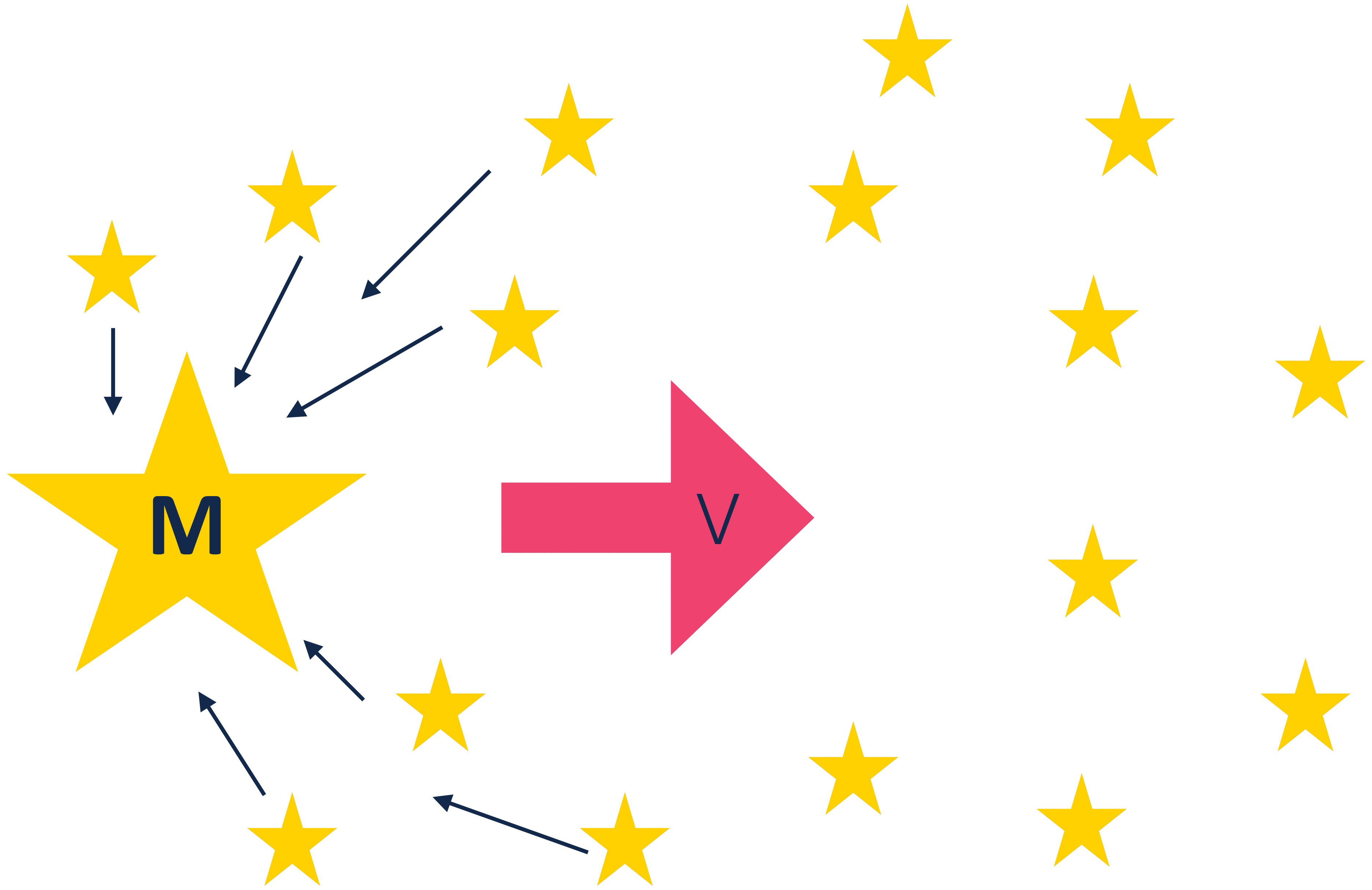

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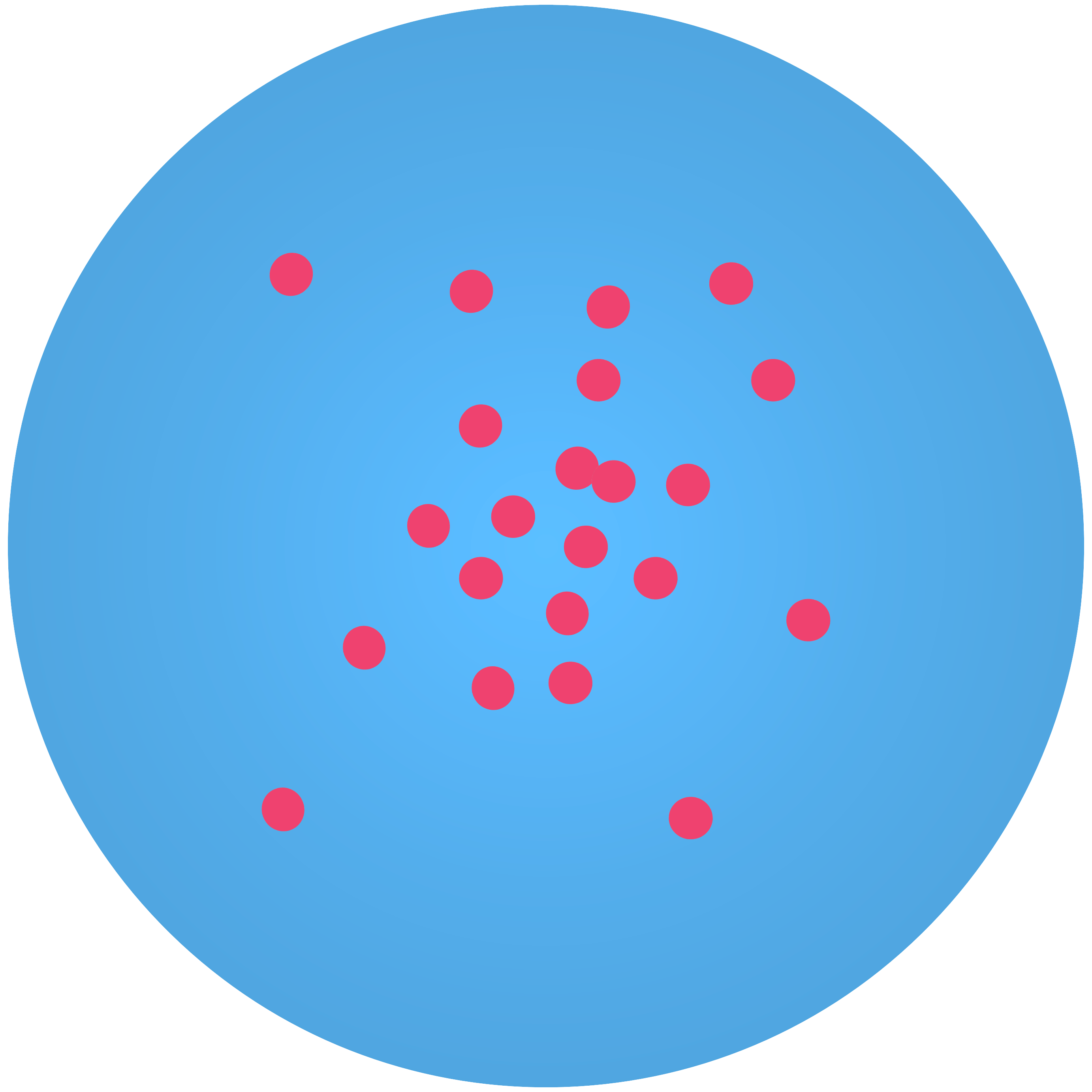

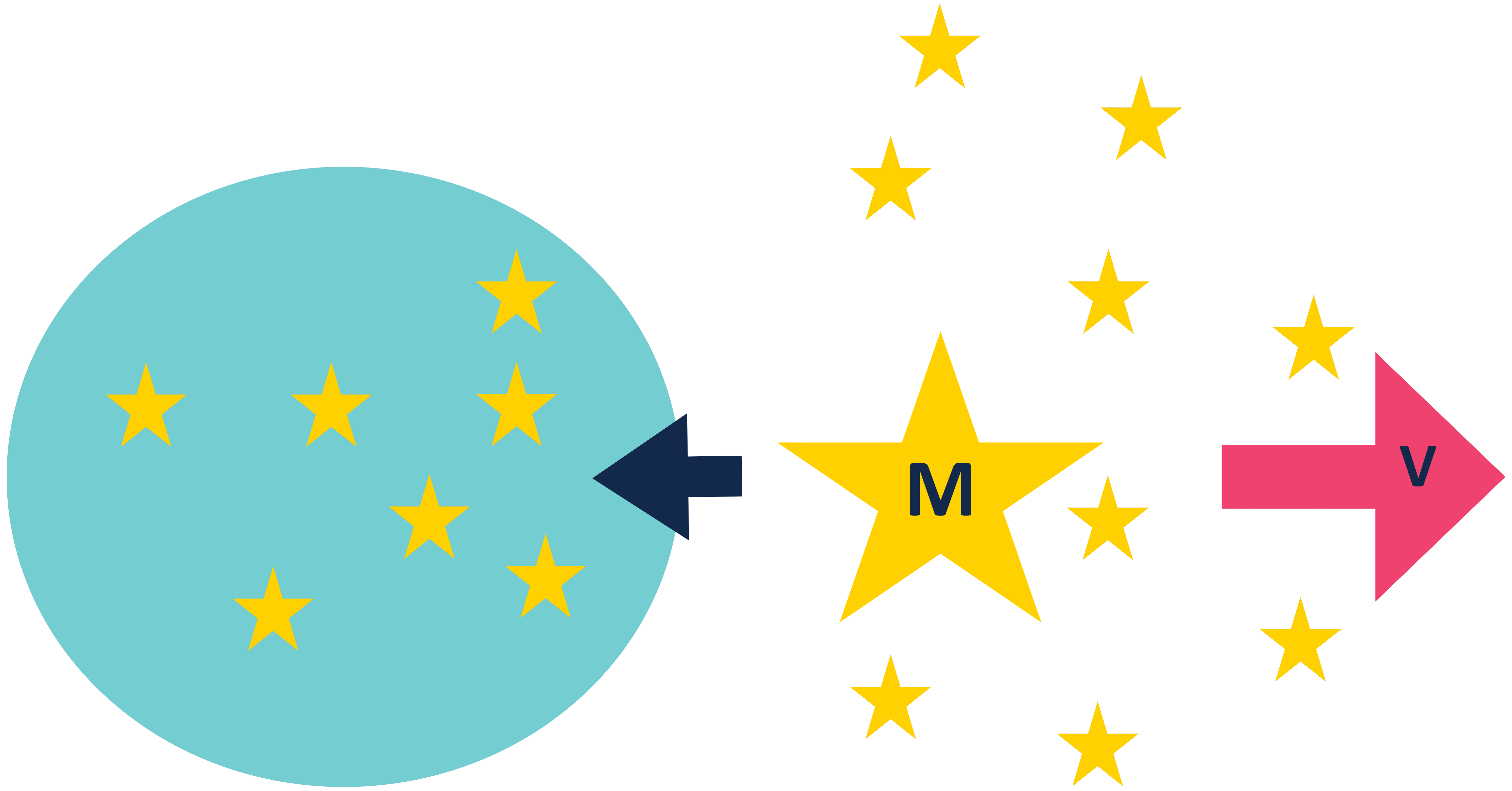

Massive particles will "**mass segregate**" into center of the cluster

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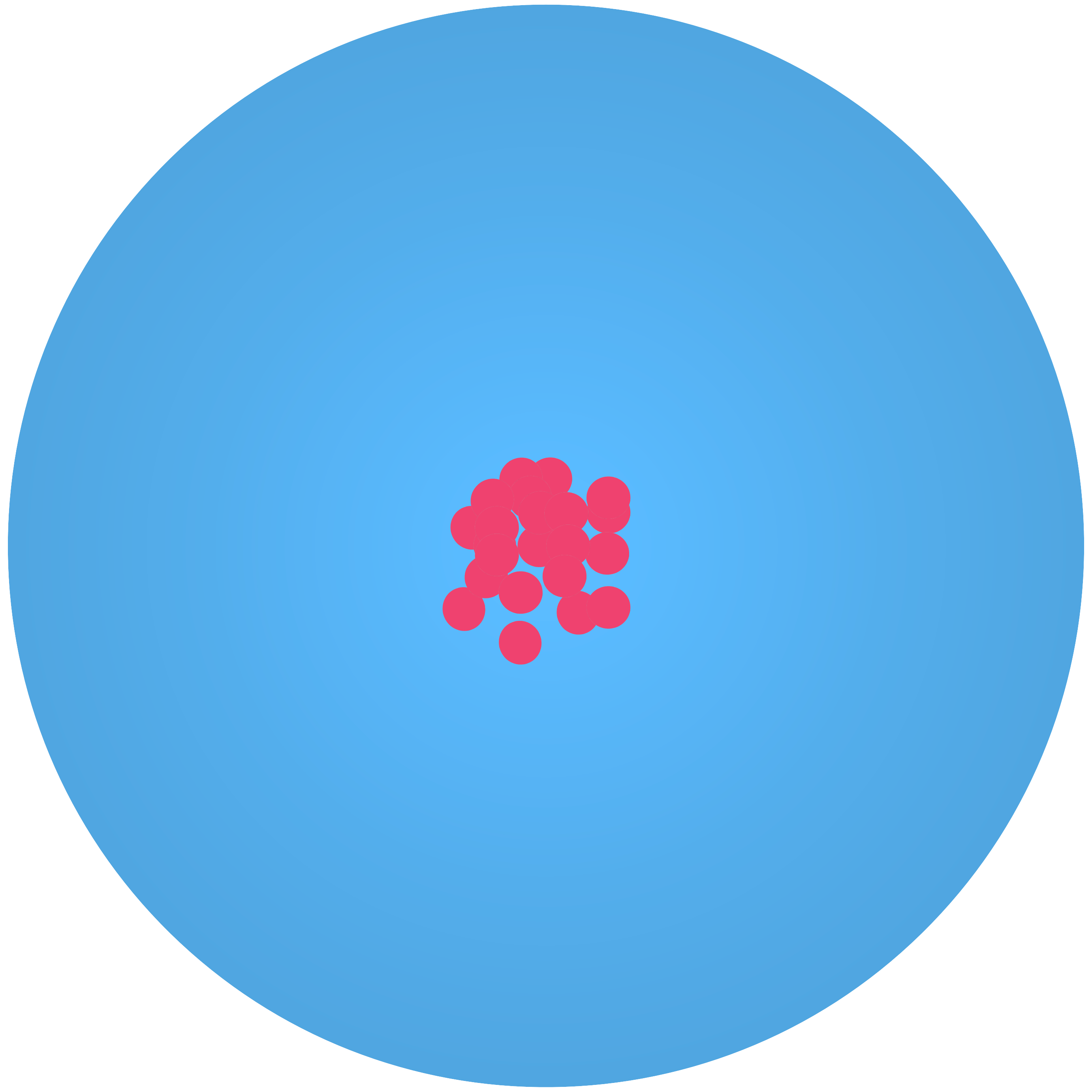
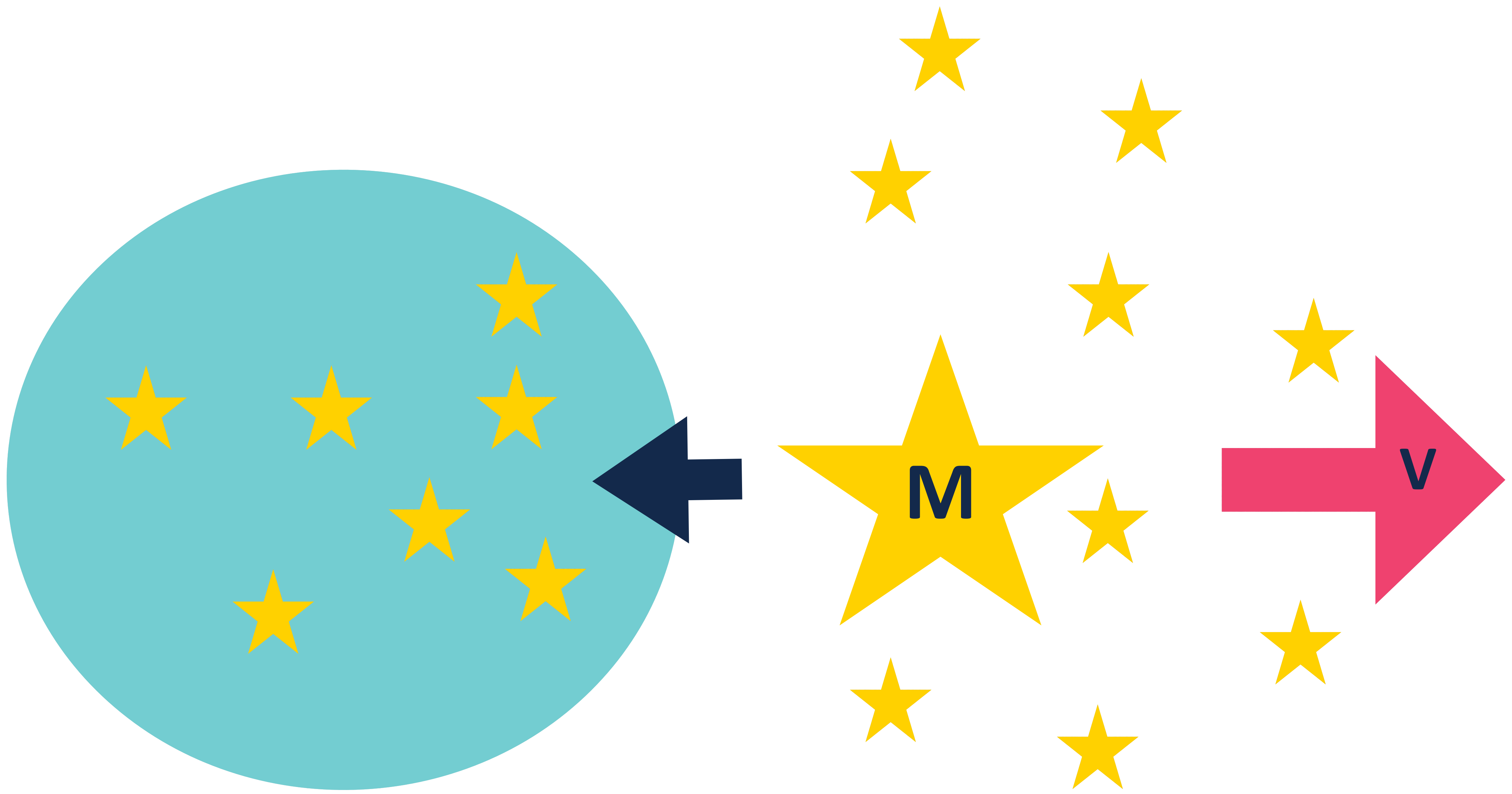
Dynamical Friction



~ 100 Myr



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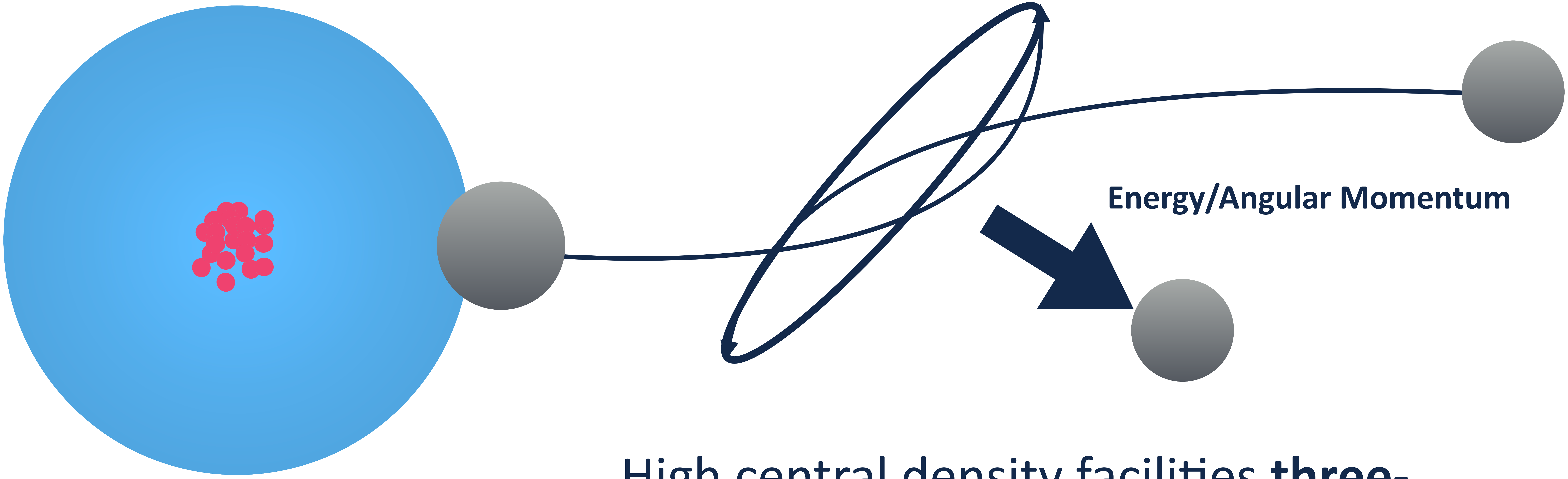
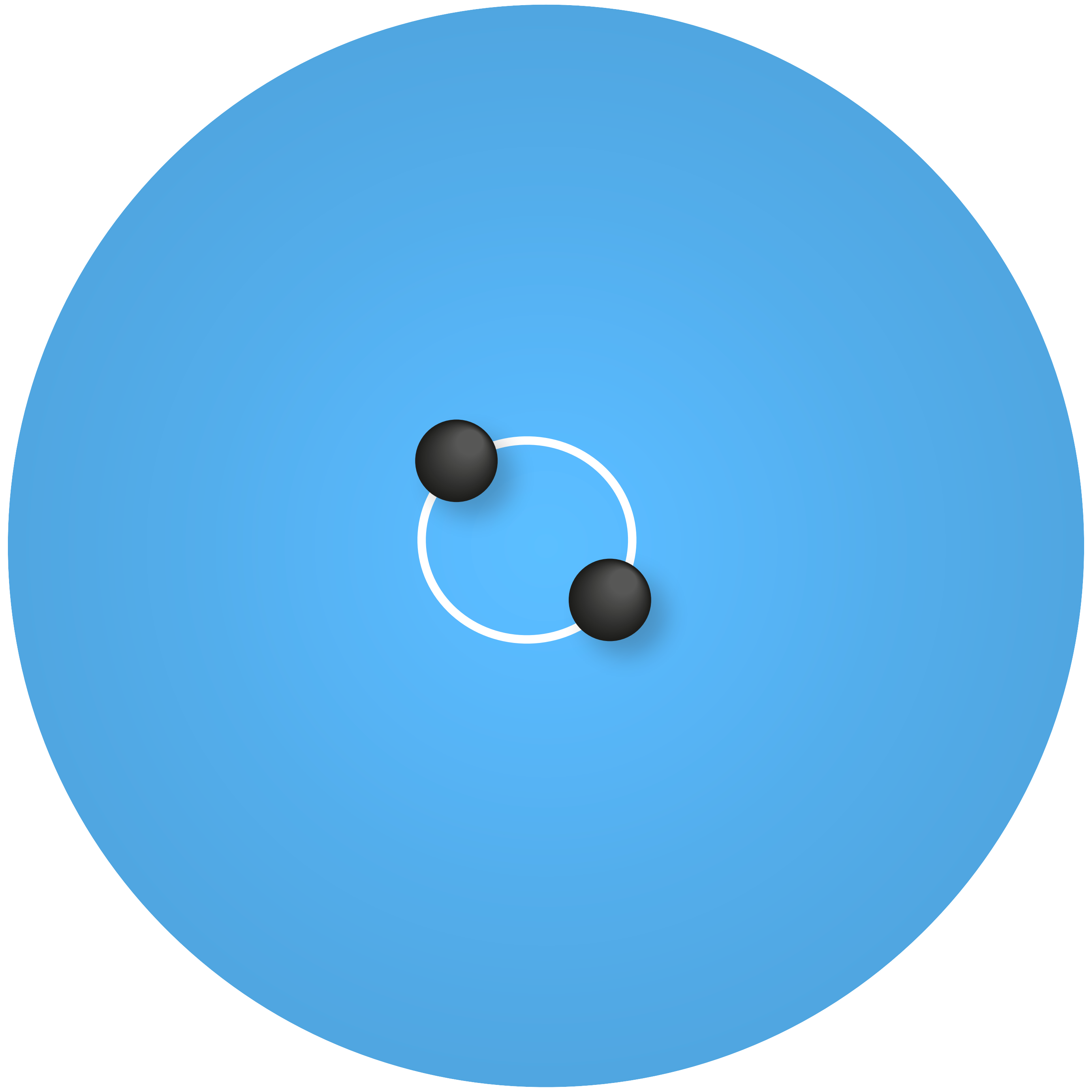
Dynamical Binary Formation



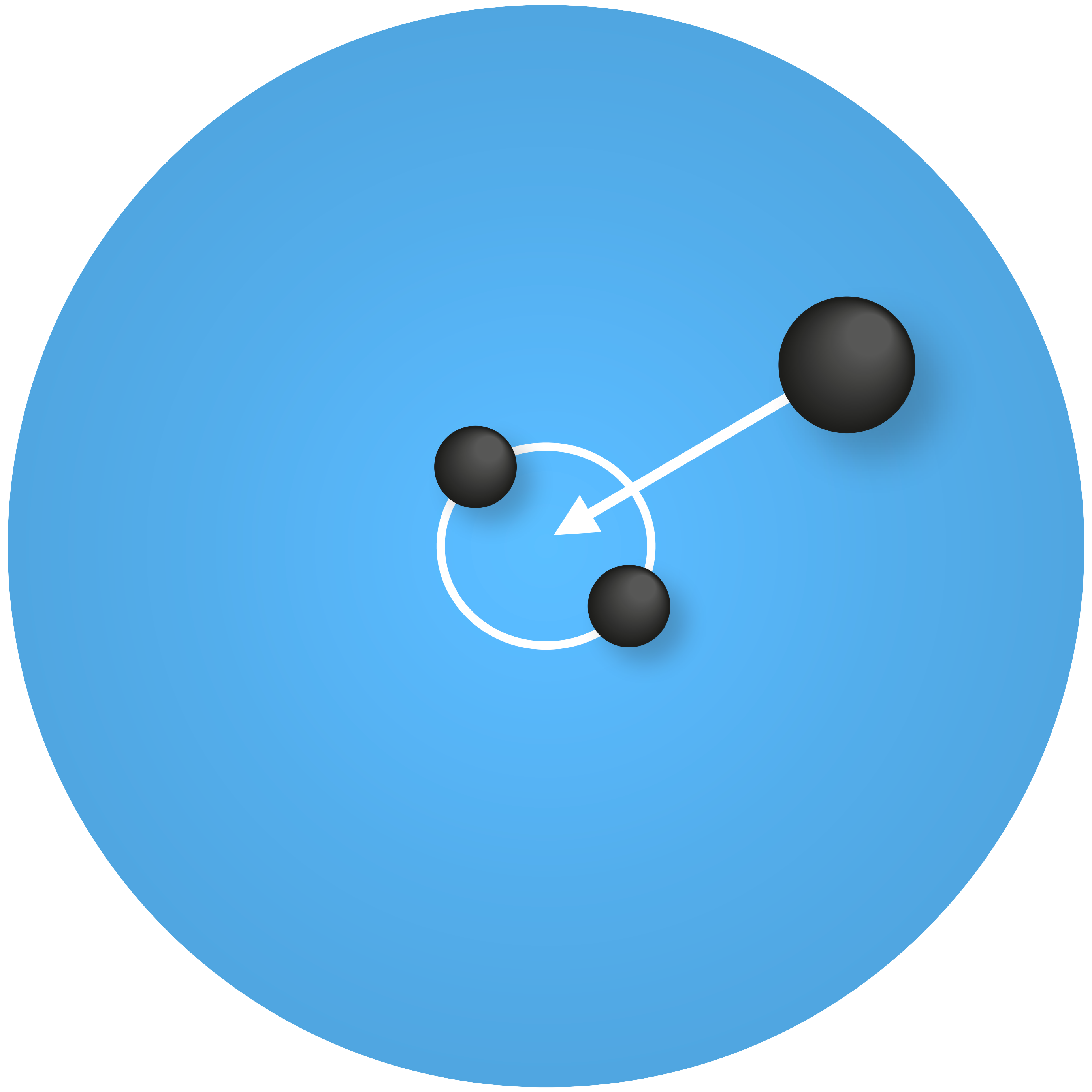
High central density facilities **three-body encounters** which form binaries from single stars and black holes

Dynamical Binary Formation



Heggie's (1975) Law

Hard binary: $E_{\rm bin} = \frac{Gm_1m_2}{2a} > \frac{1}{2}\langle mv^2 \rangle$



On average, three-body encounters **harden hard binaries** and **soften soft binaries**

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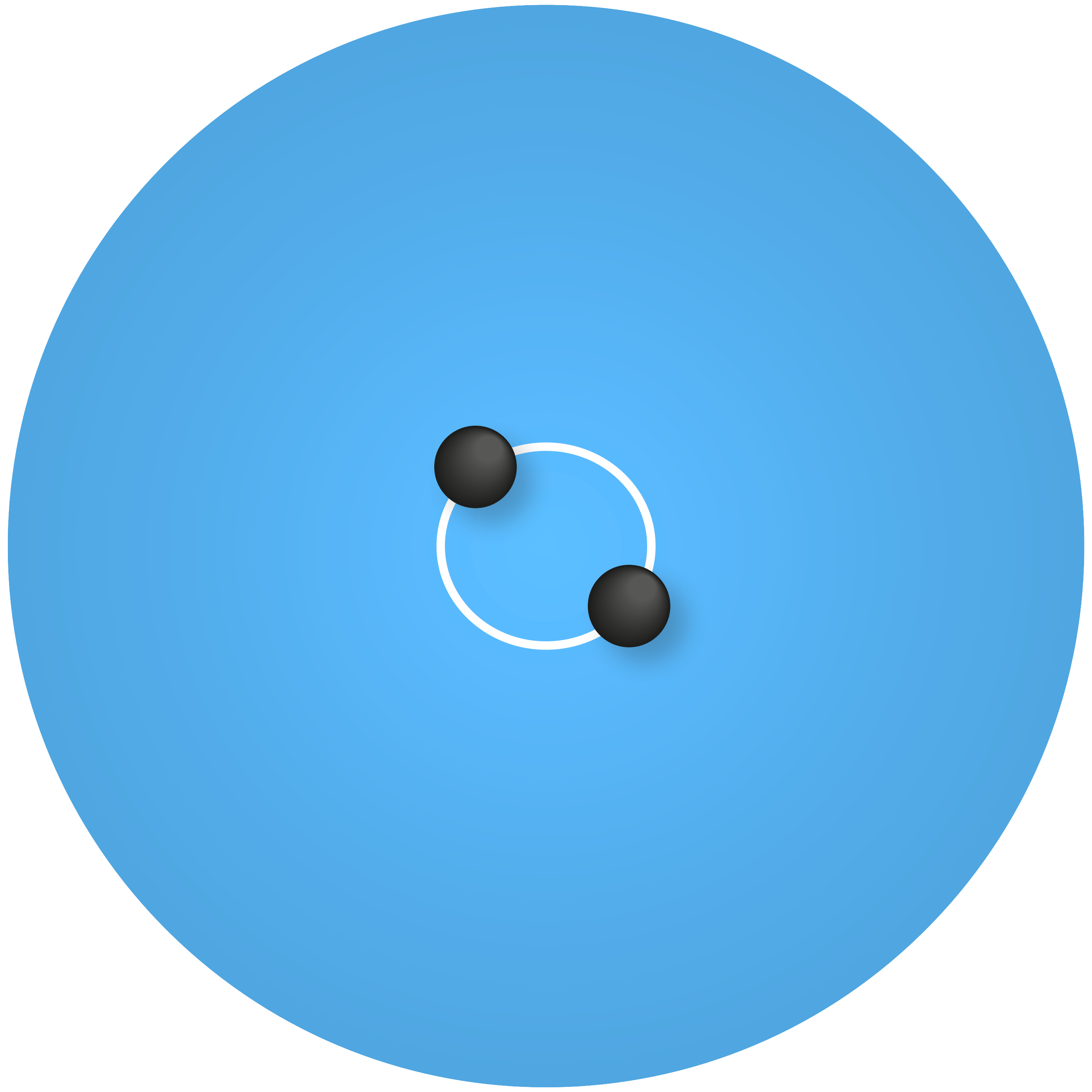

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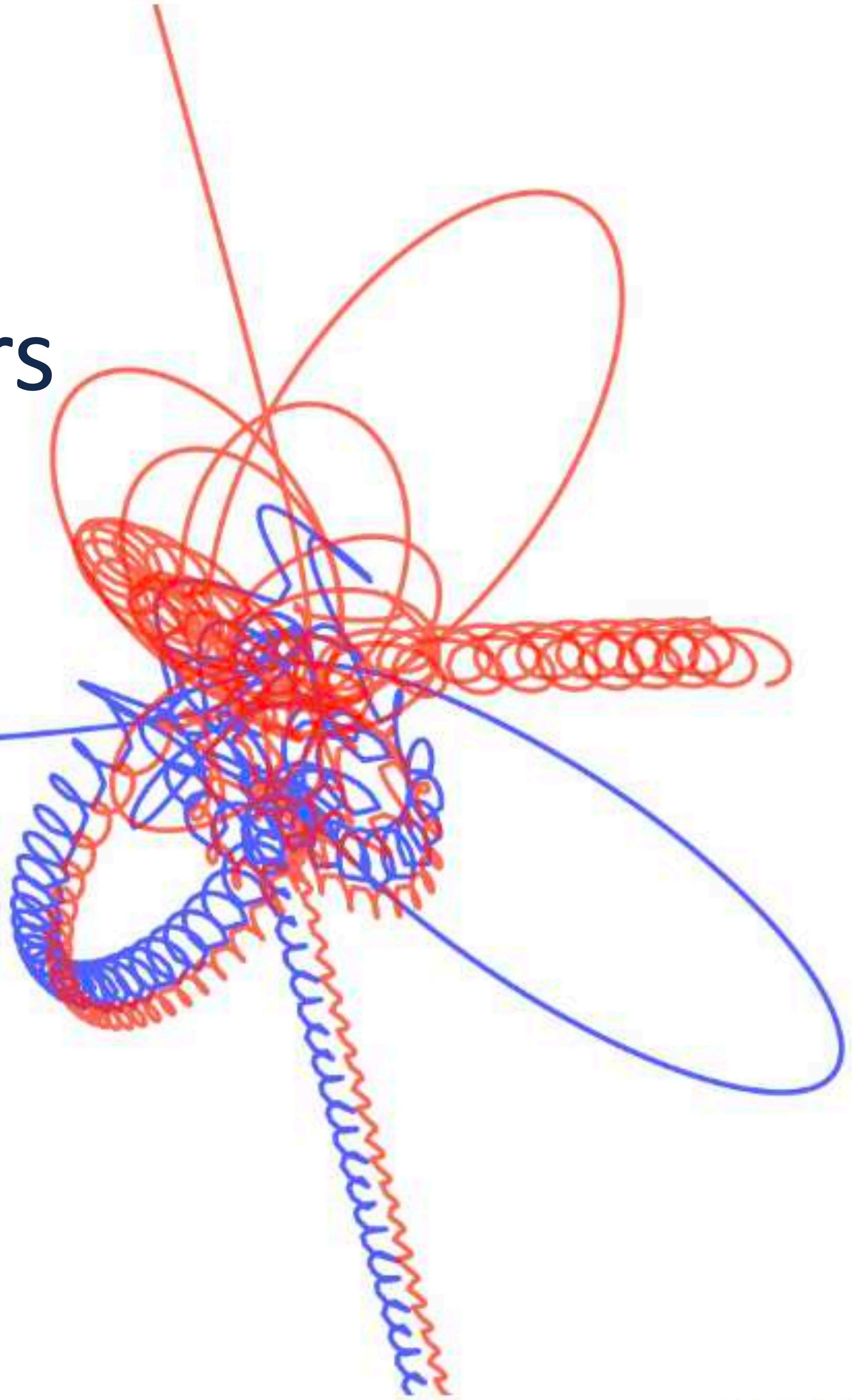

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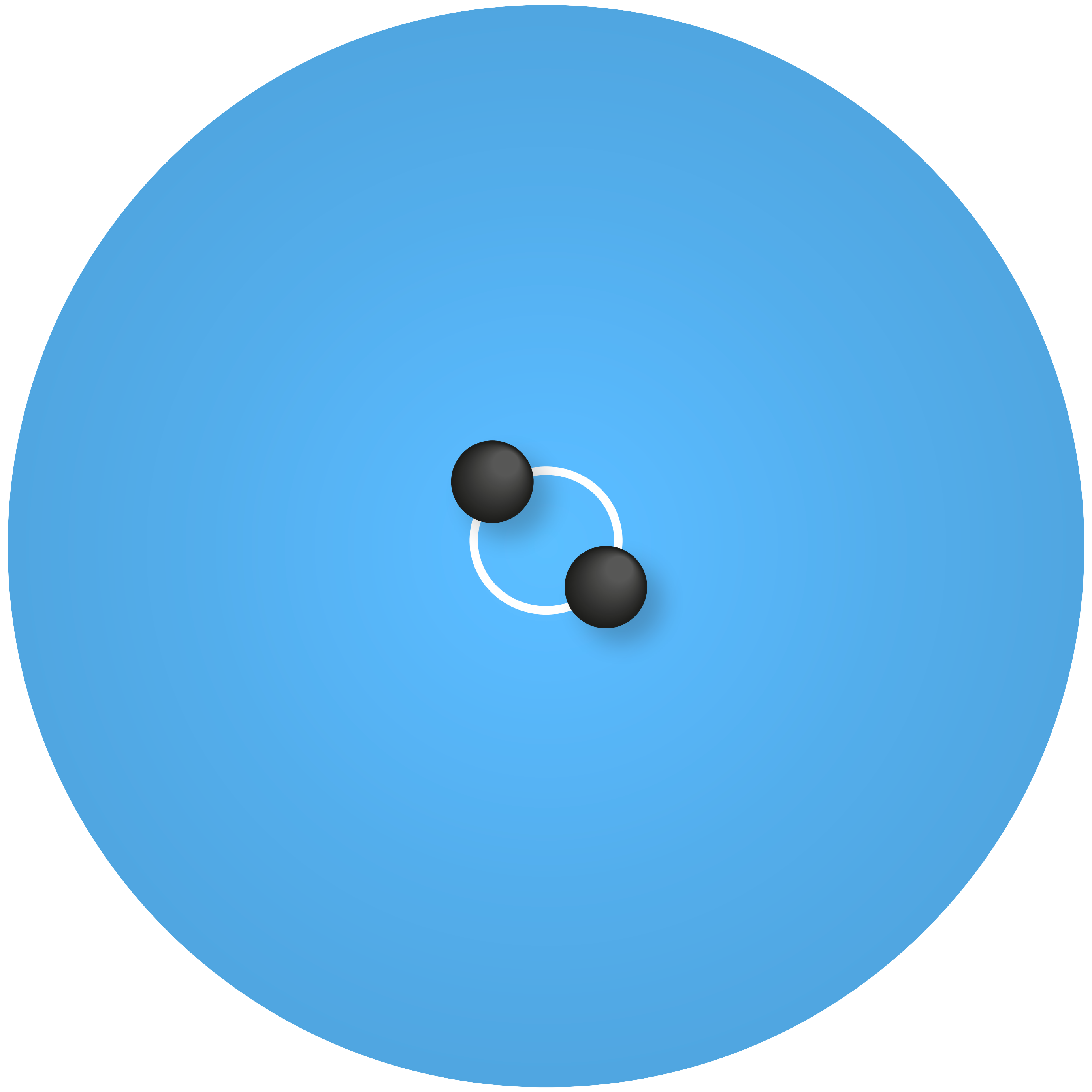

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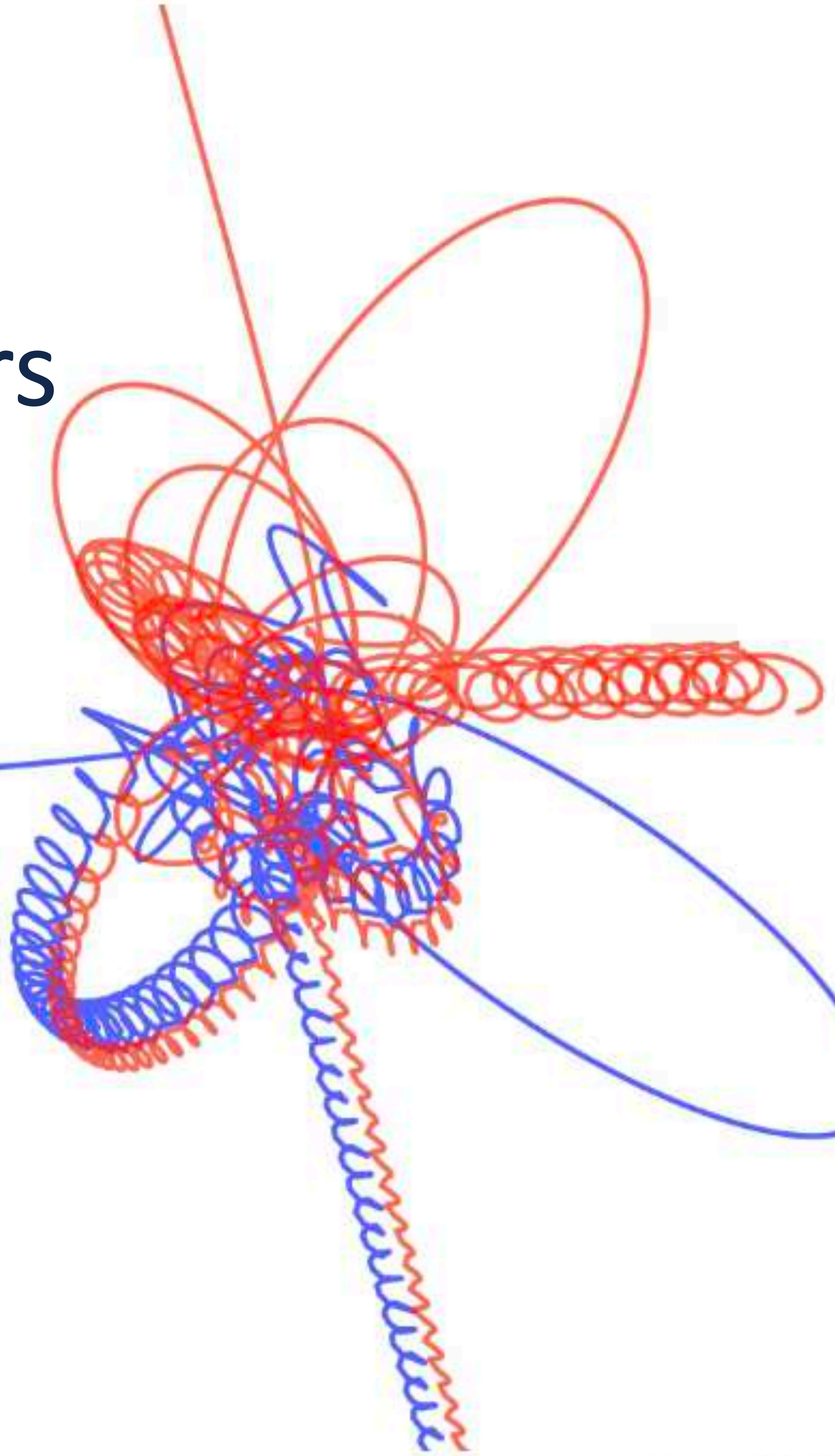

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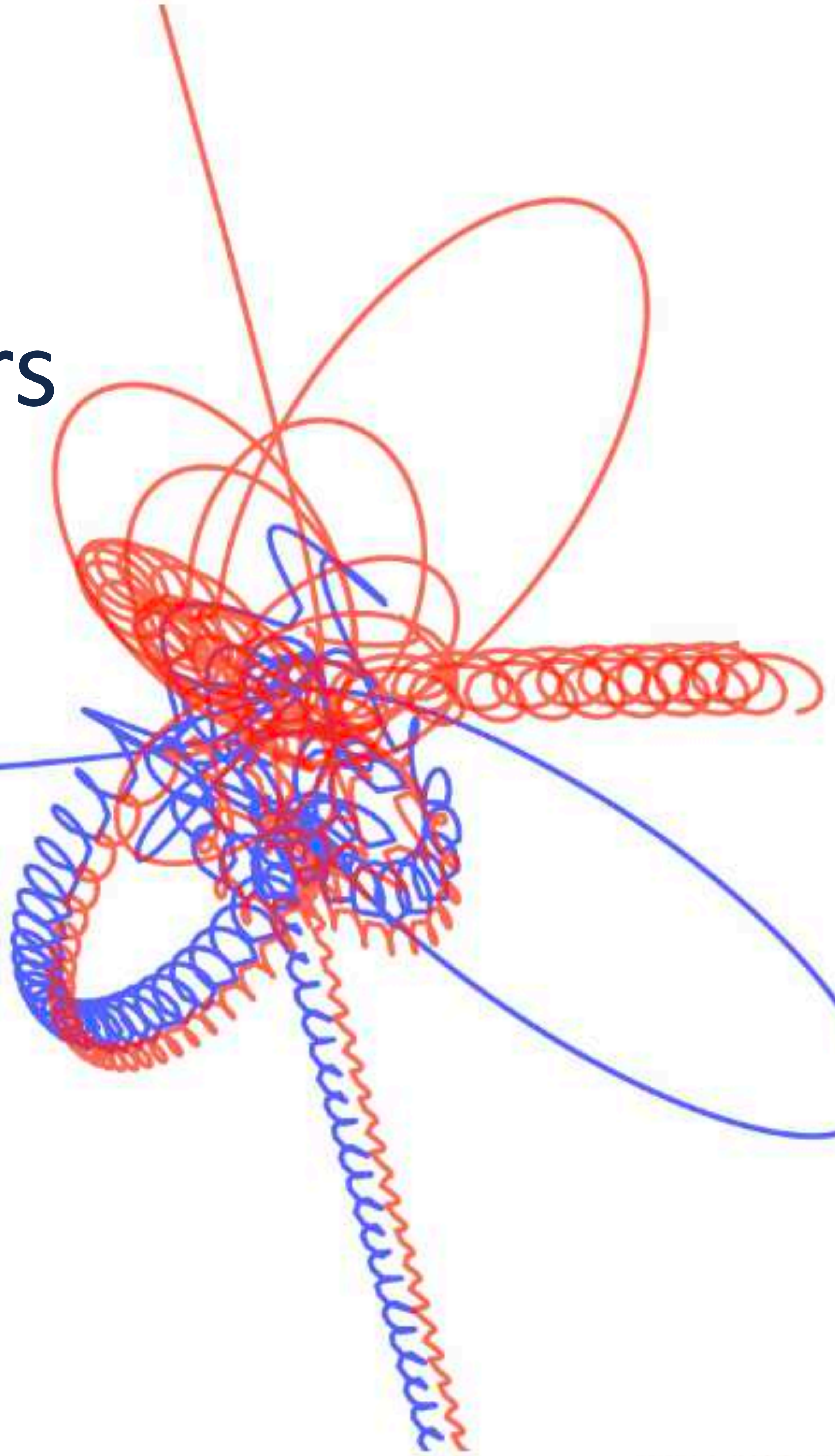

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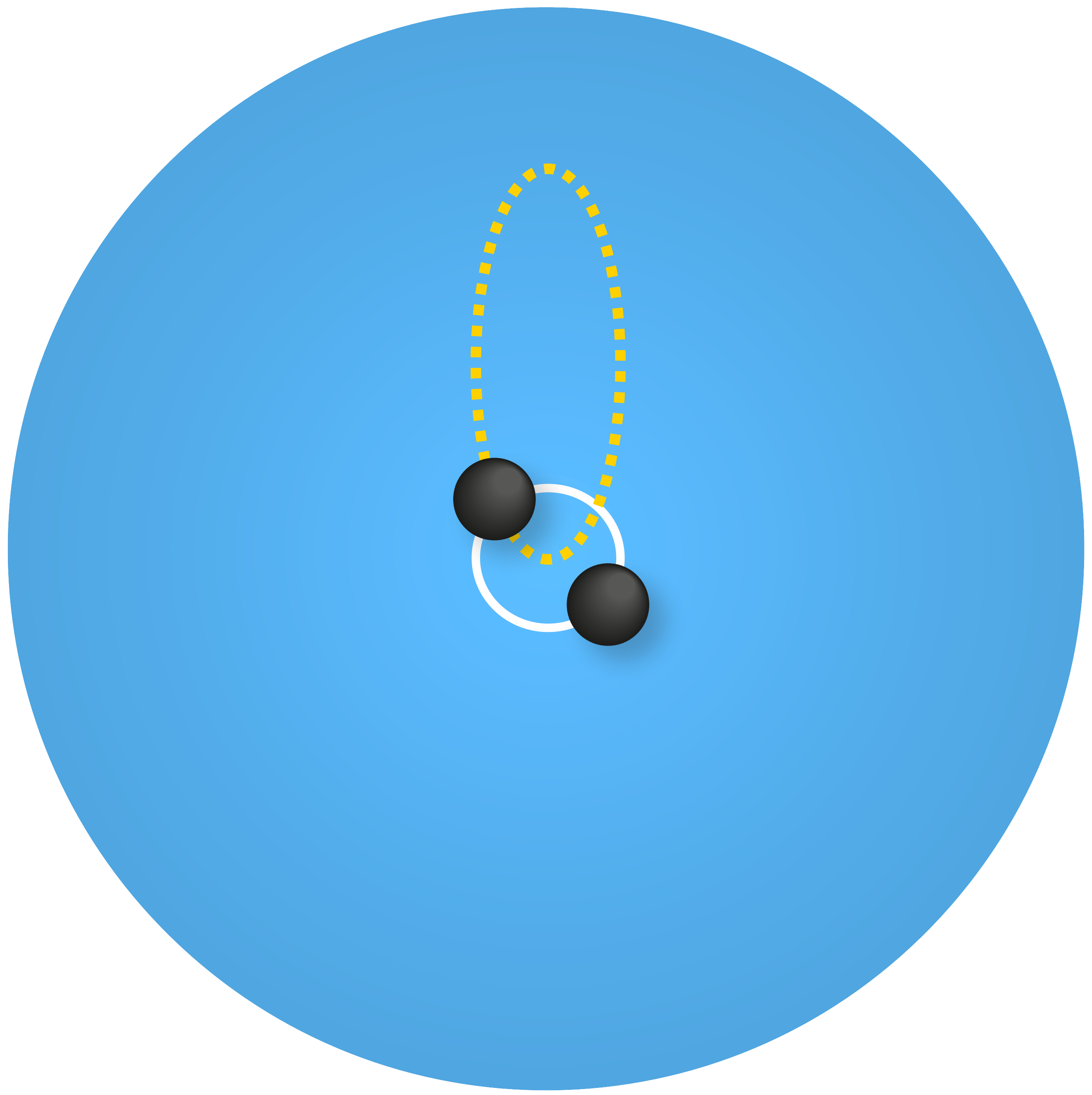

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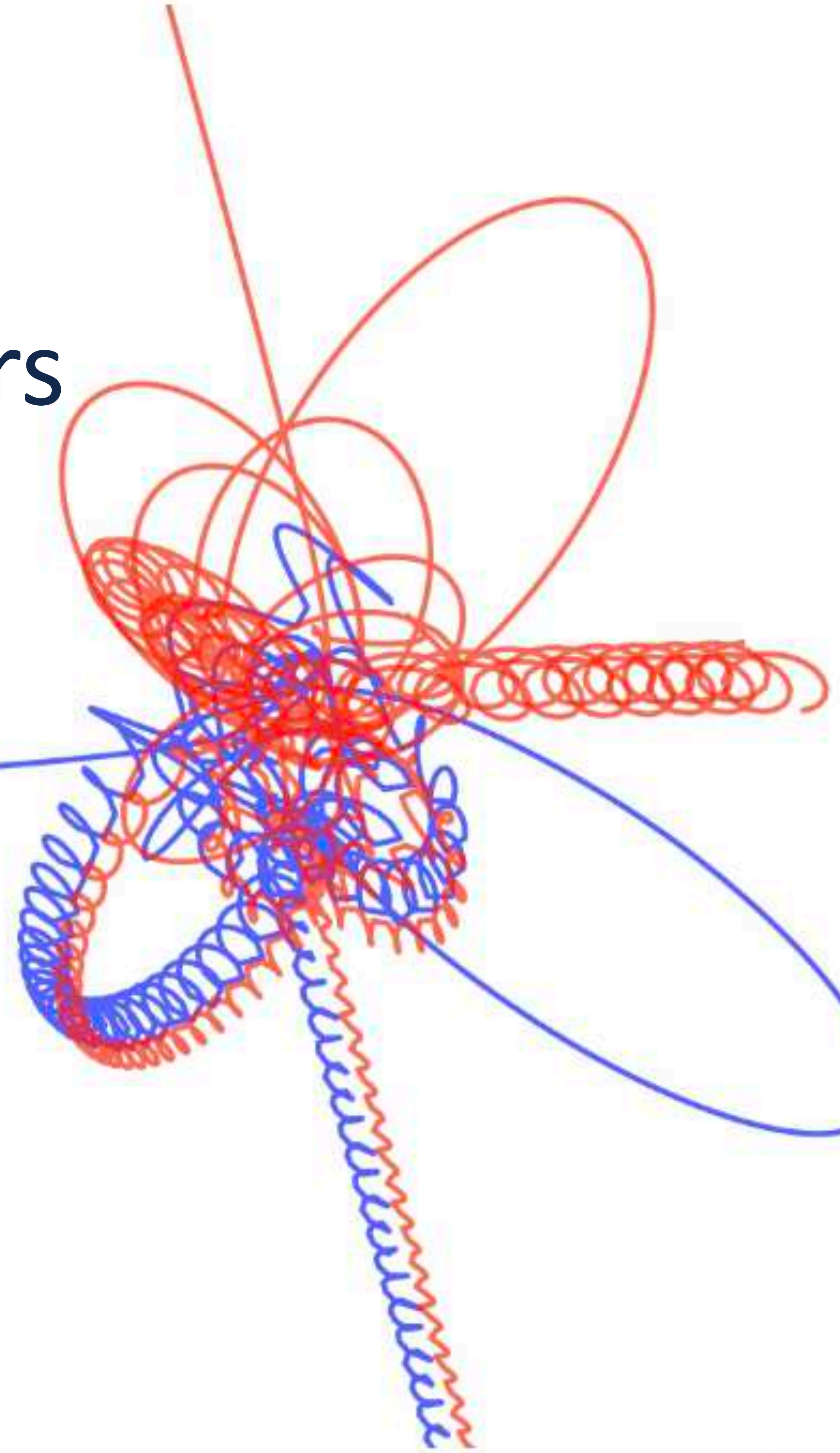

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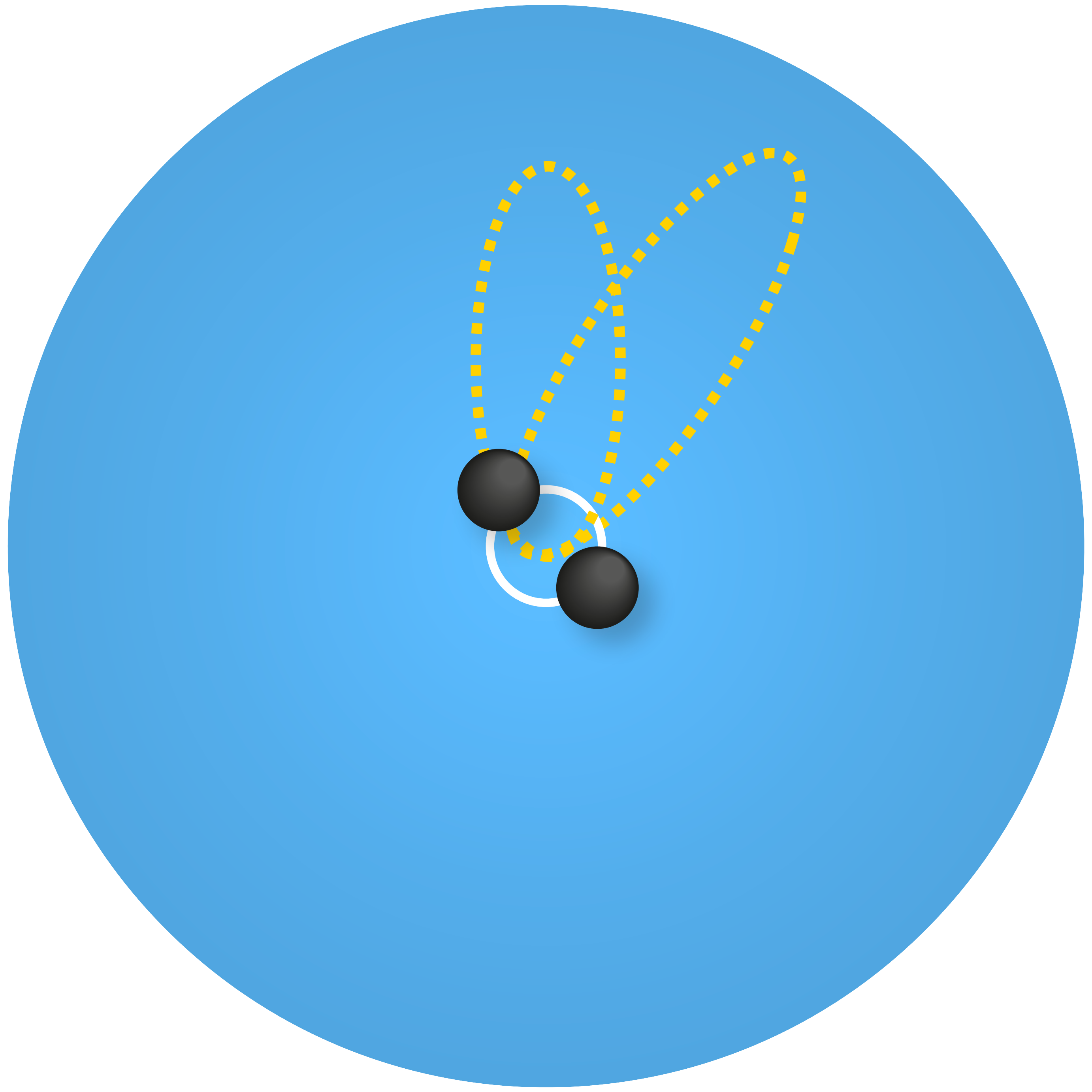

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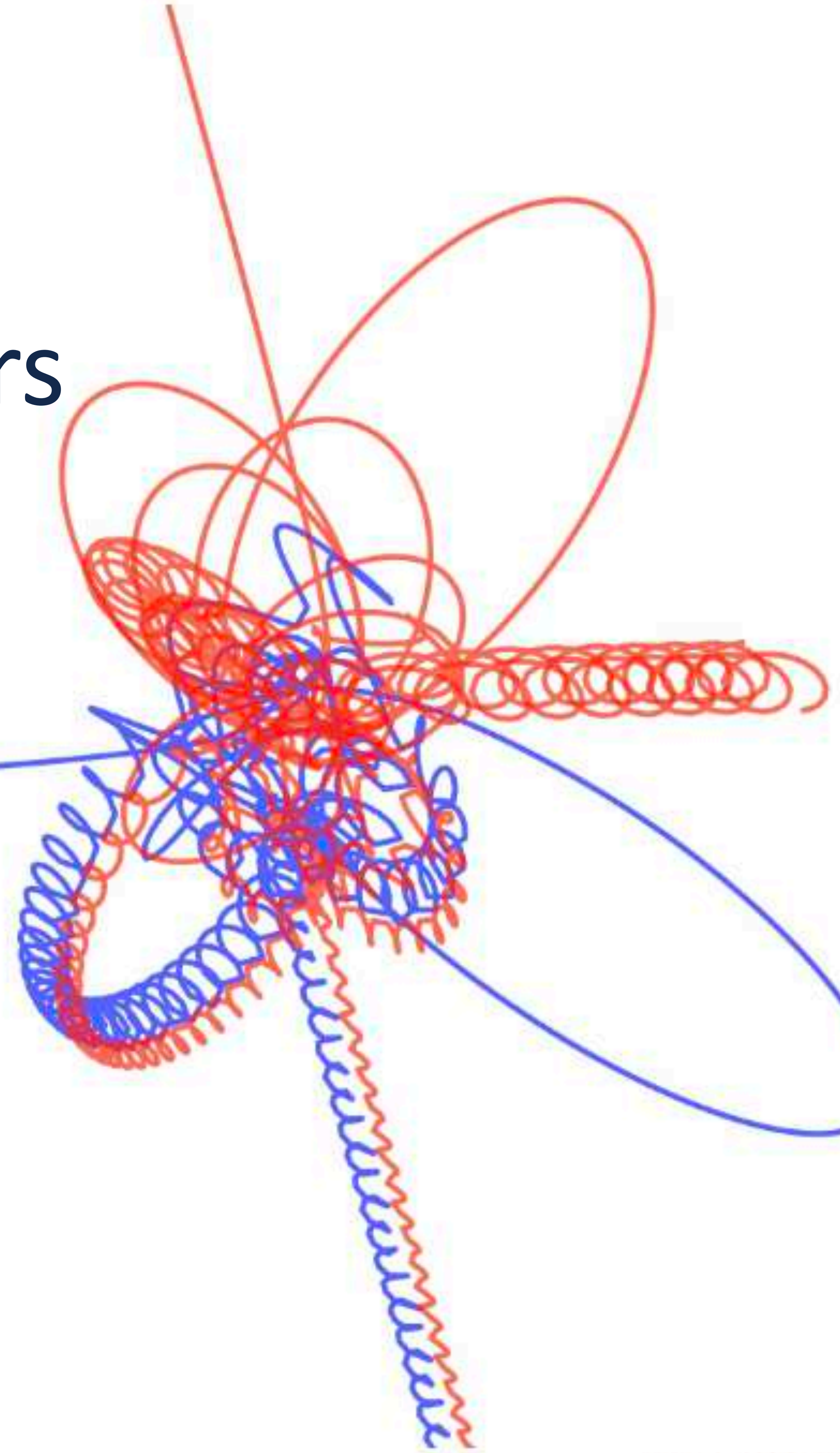

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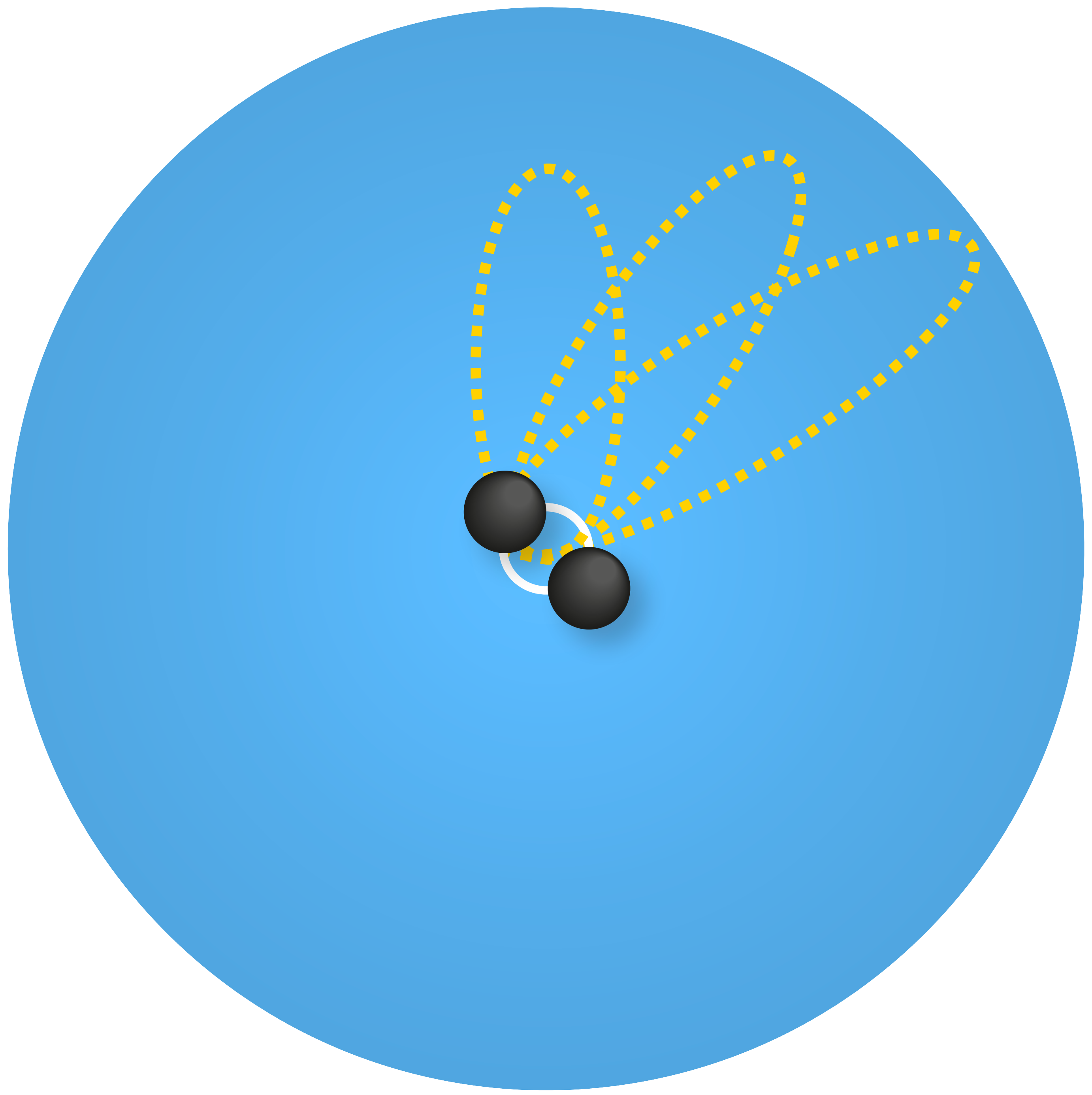

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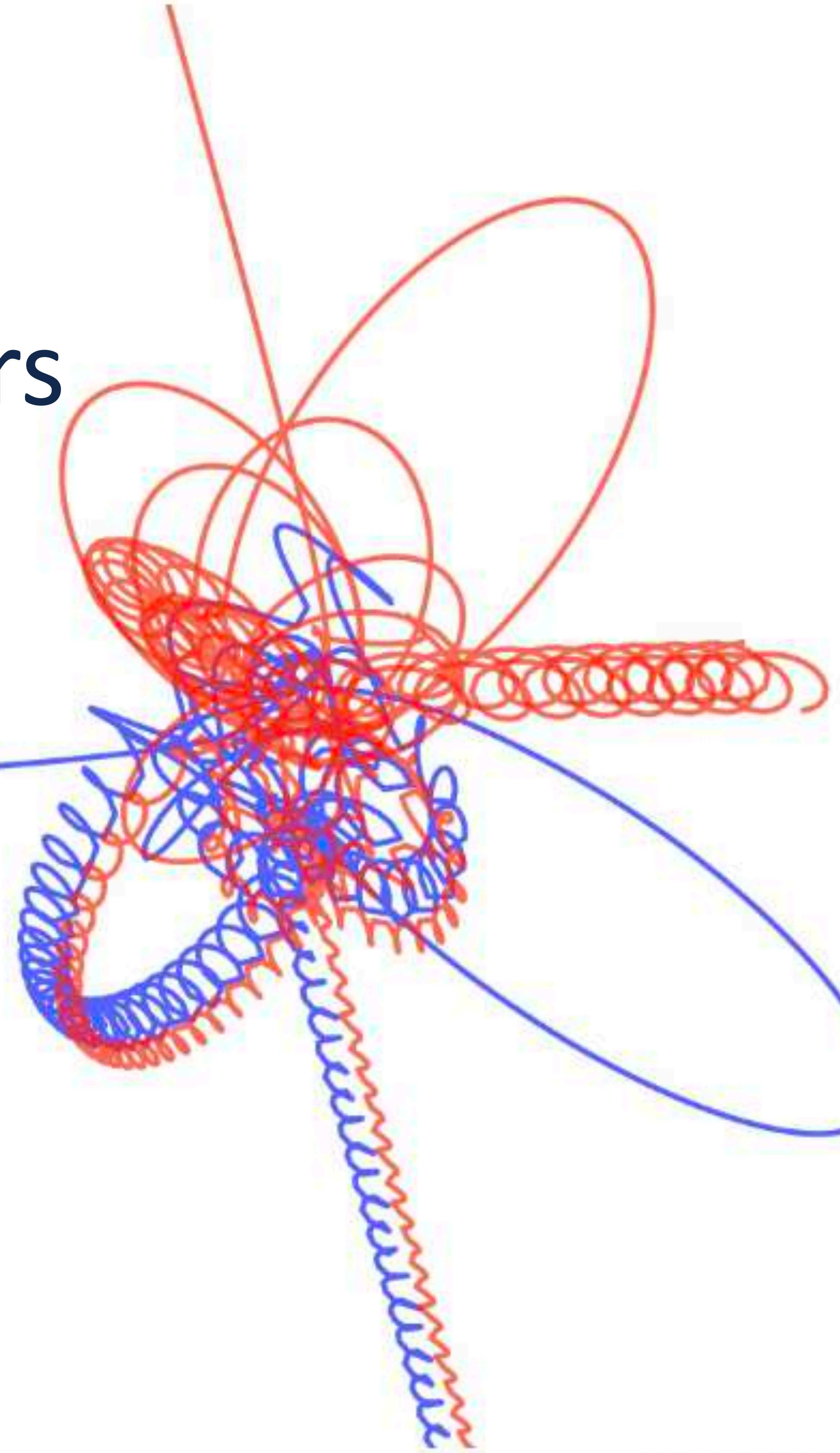

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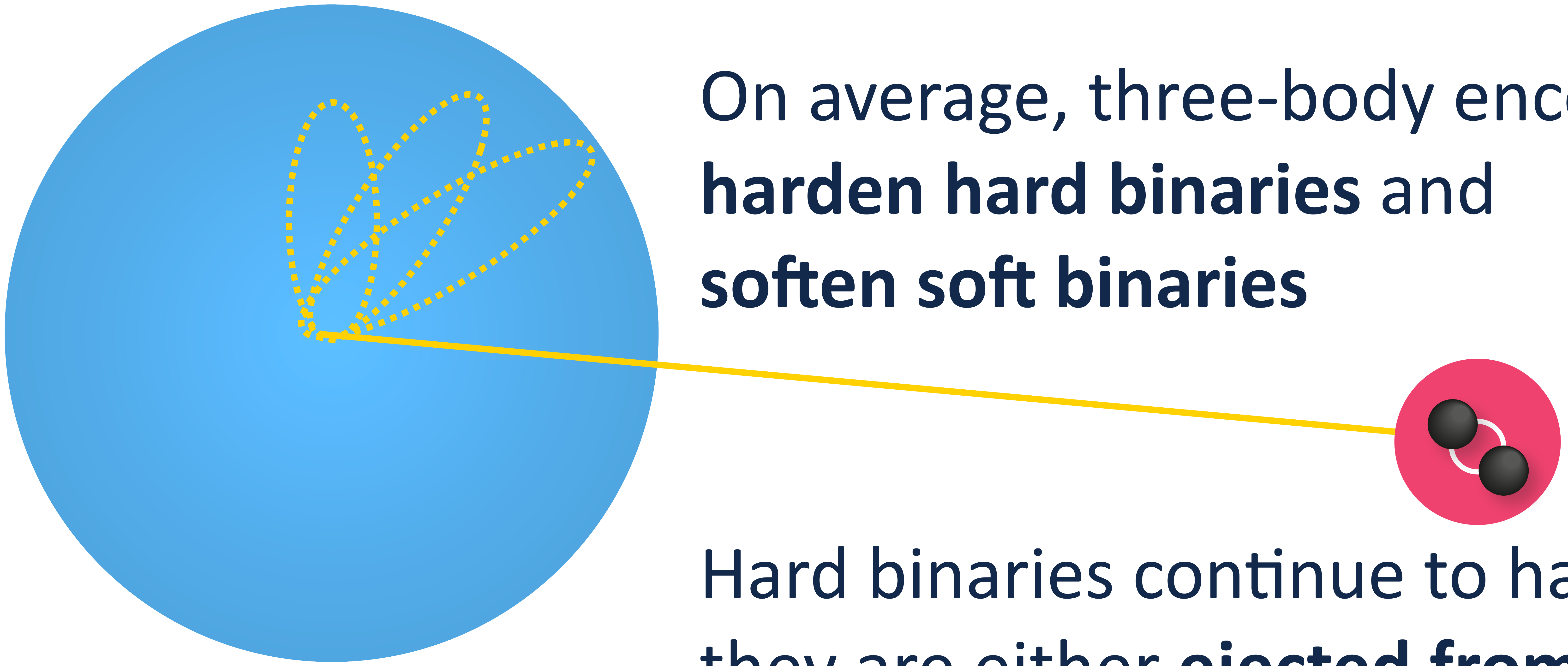

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Hard binaries continue to harden until they are either **ejected from the cluster**

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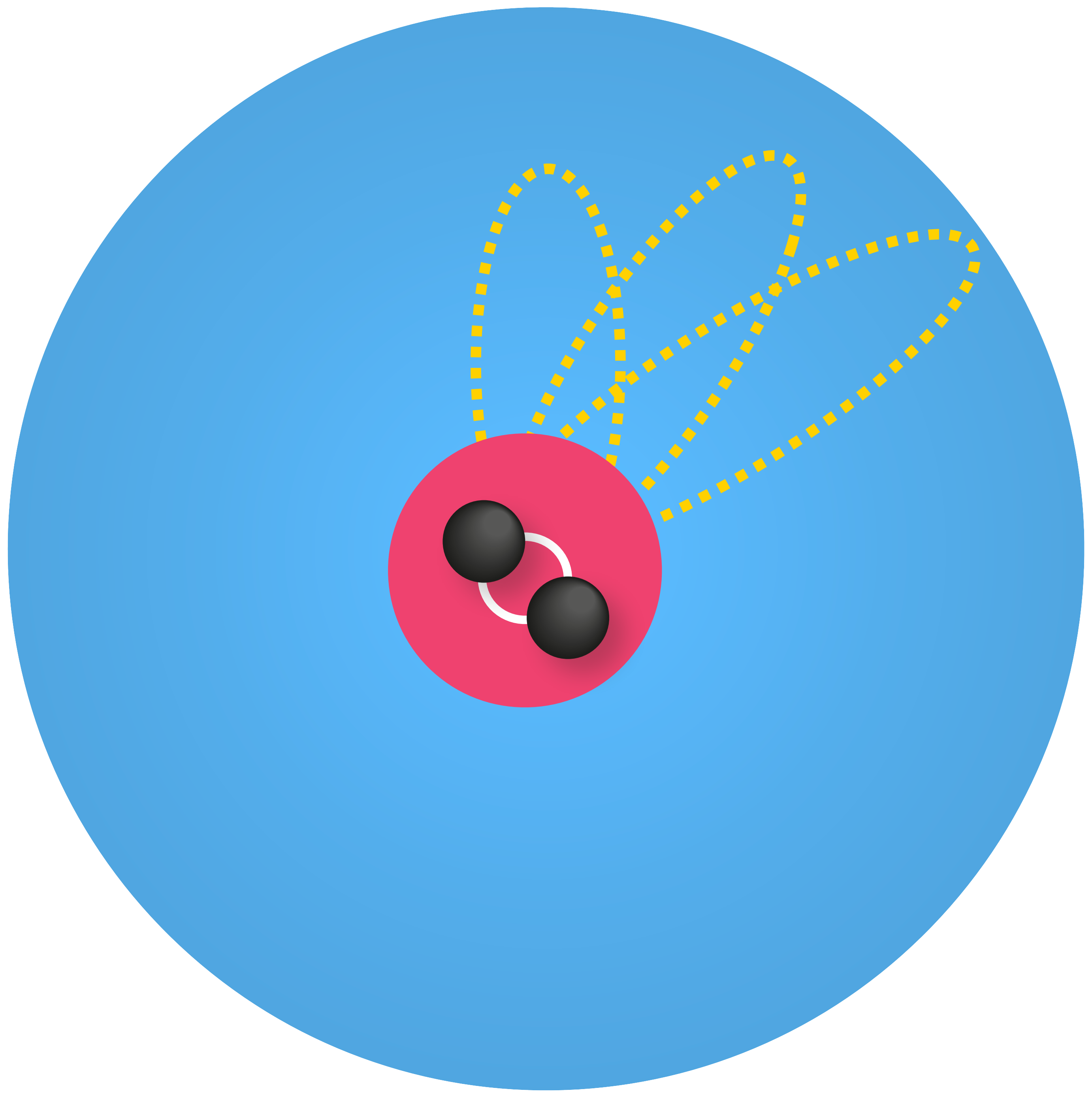

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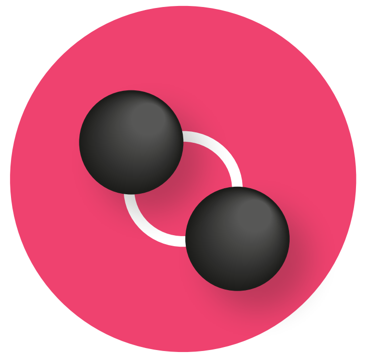

On average, three-body encounters **harden hard binaries** and **soften soft binaries**

A single hard binary can have the same binding energy as the entire cluster!

Hard binaries continue to harden until they are either **ejected from** the cluster, **merge inside** the cluster, or the cluster **disrupts** around them

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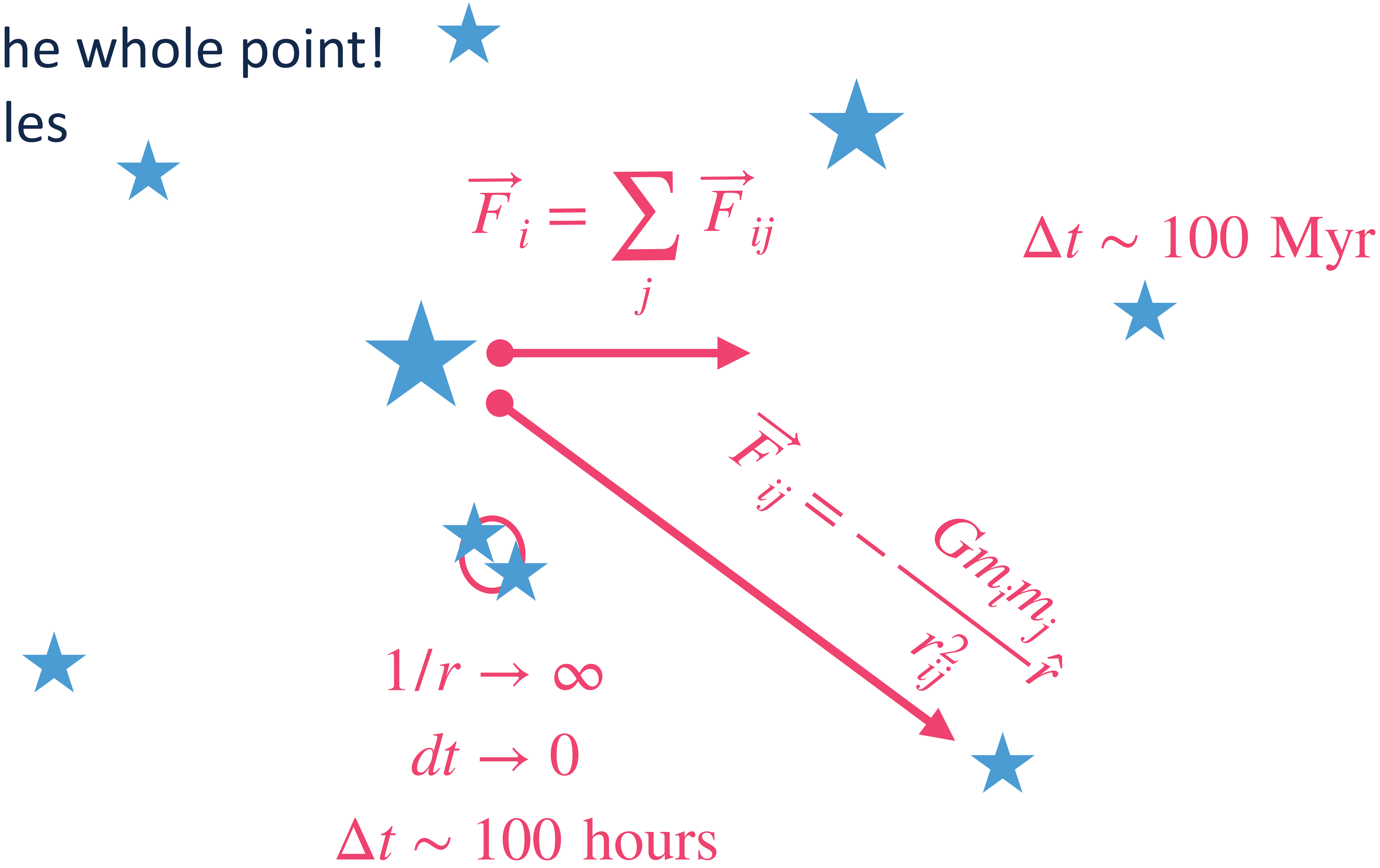
Gravitational *N*-body Problem

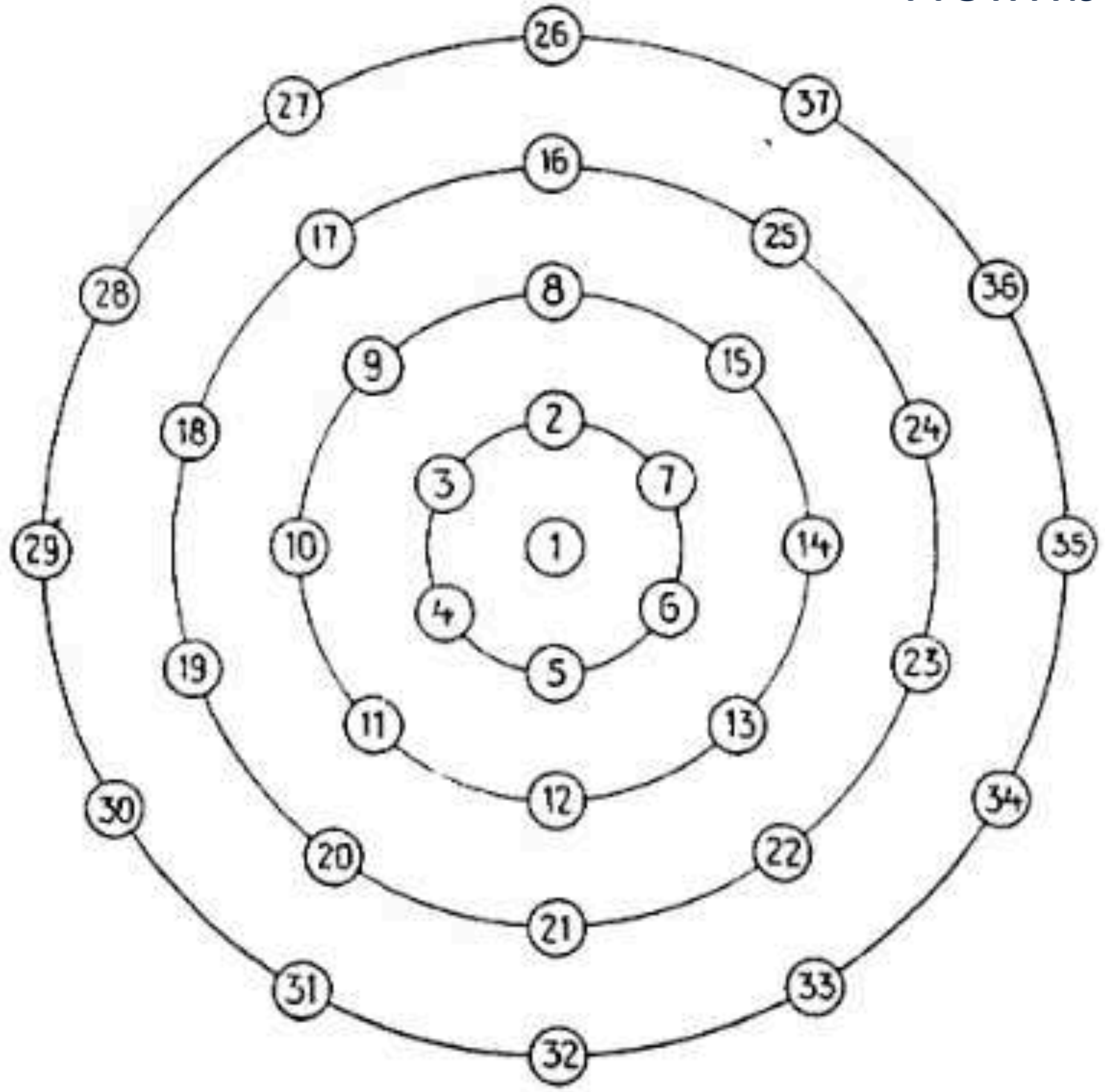
Core issues:

1. Close encounters are the whole point!
2. Wide range of timescales
3. Scales as N^2



The First *N*-body Integrators

Holmberg (1941)
ApJ 94, 3



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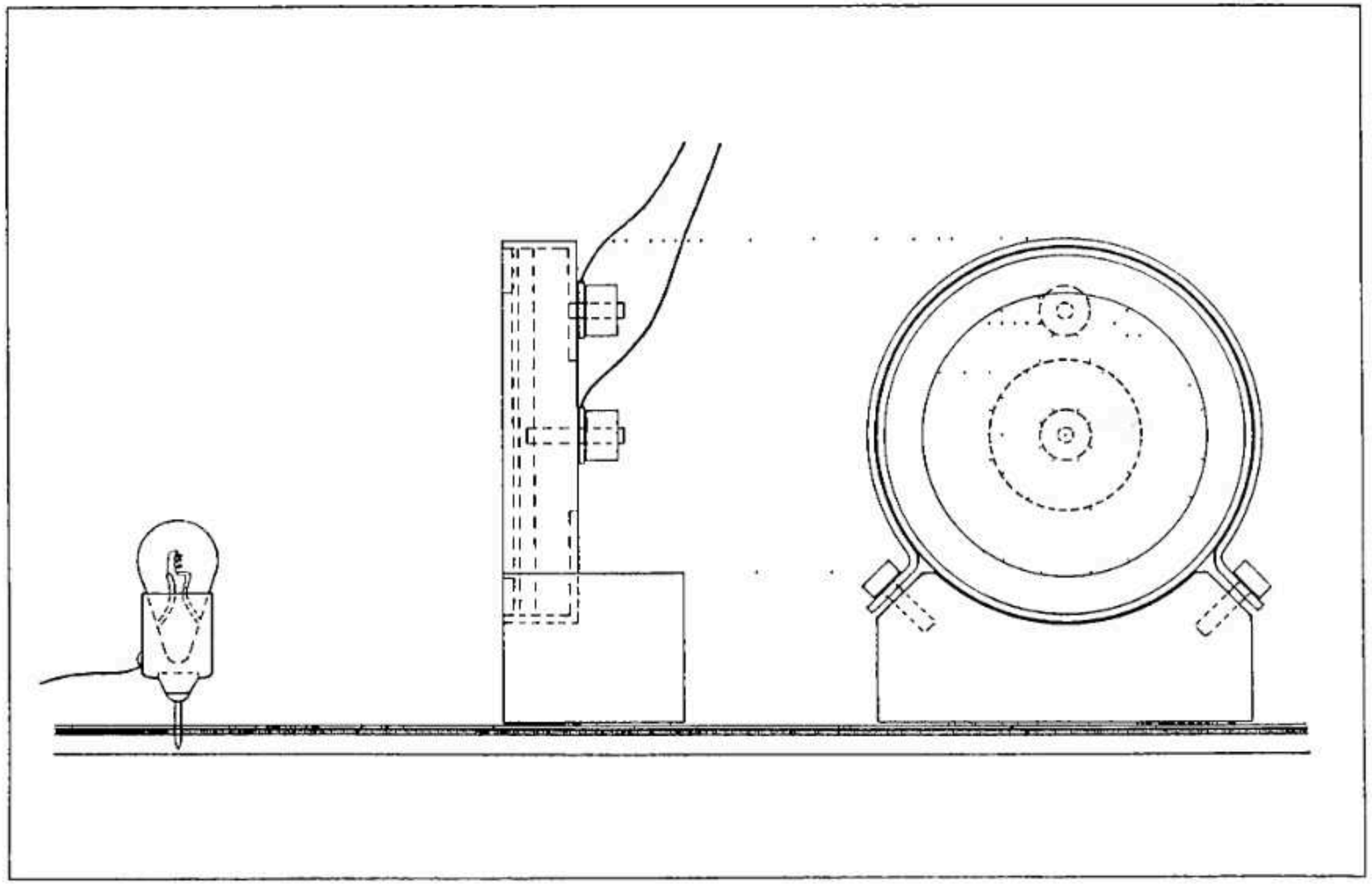
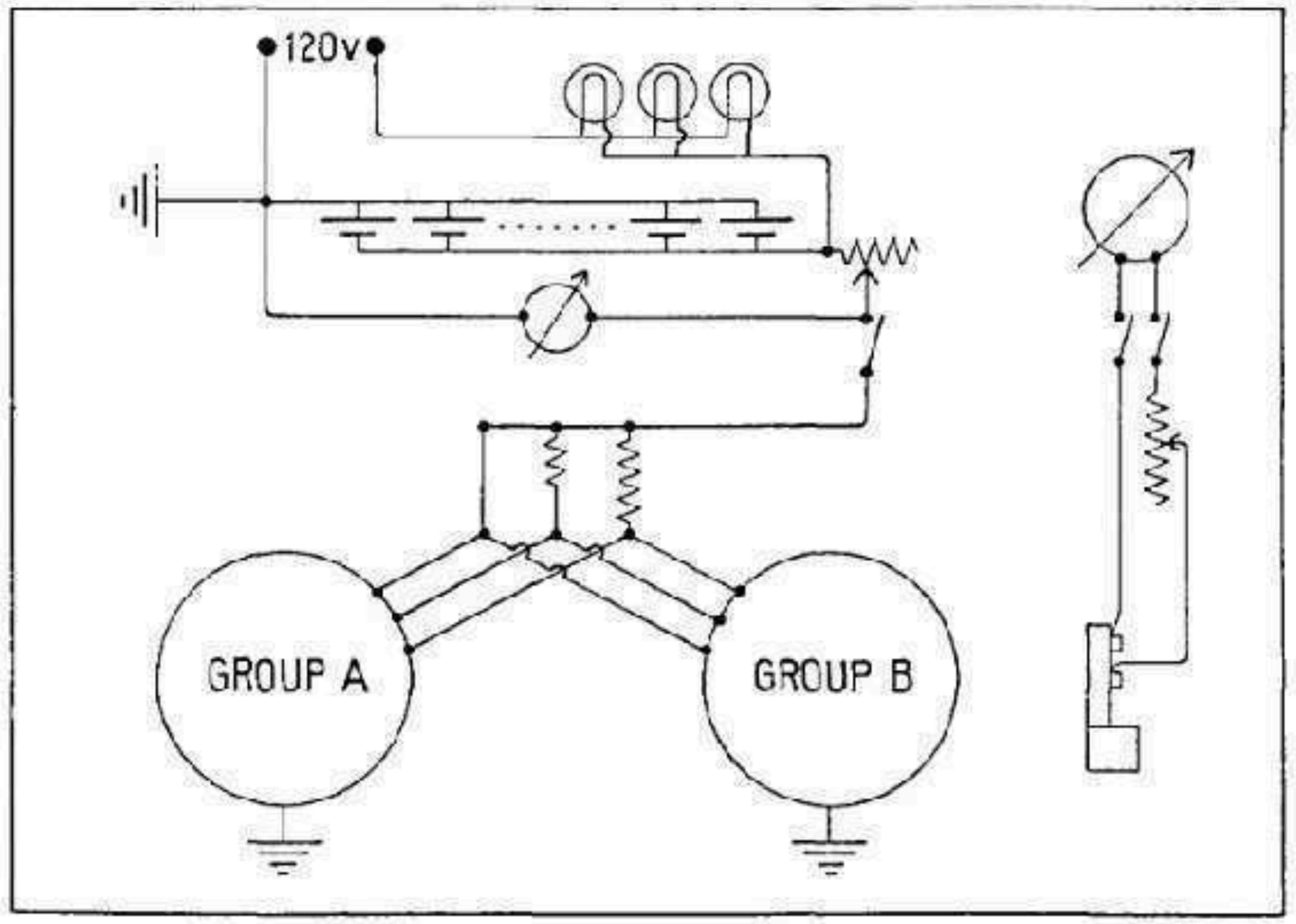
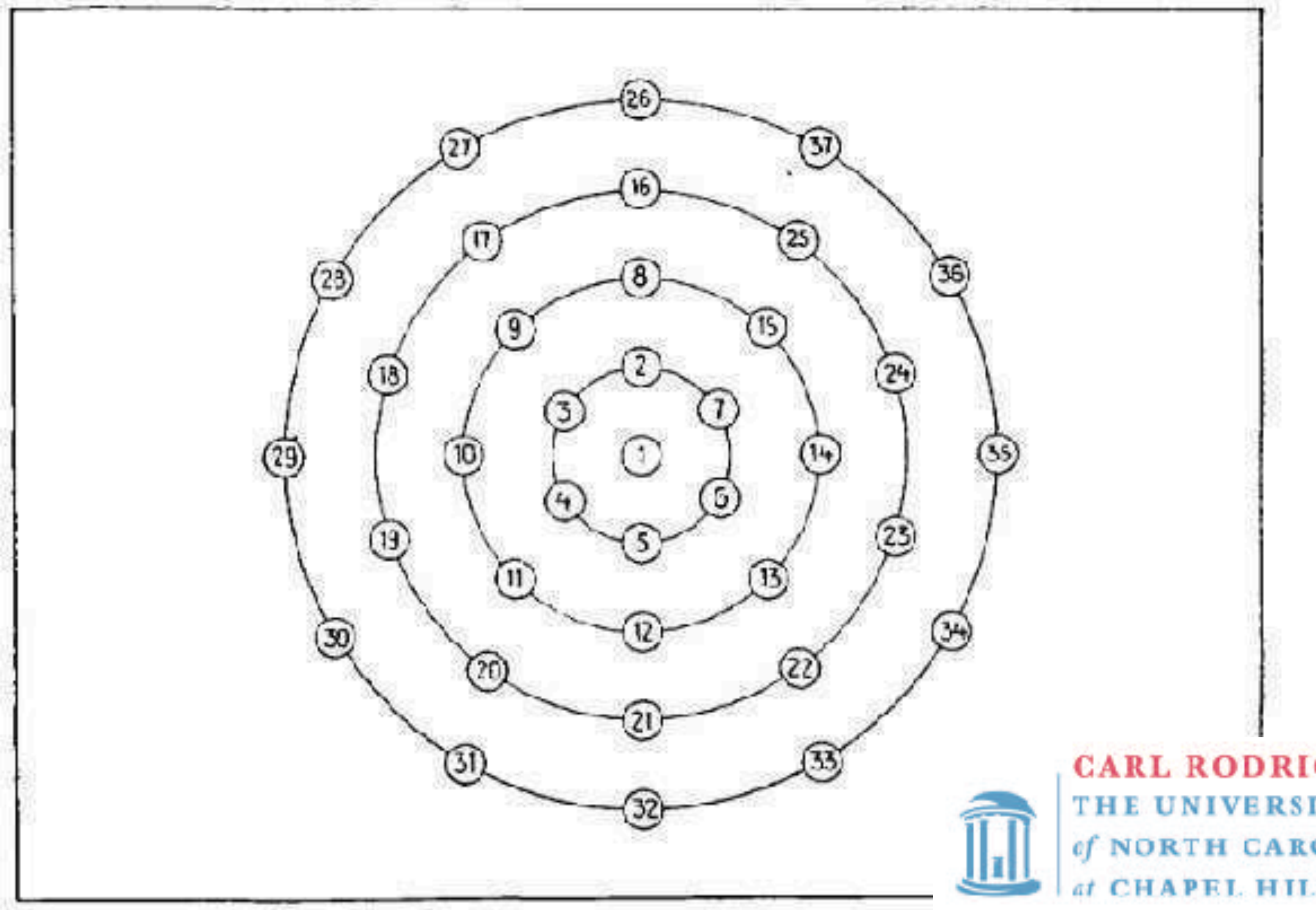


FIG. 1.—Cross-section of light-bulb and photocell (half-size)





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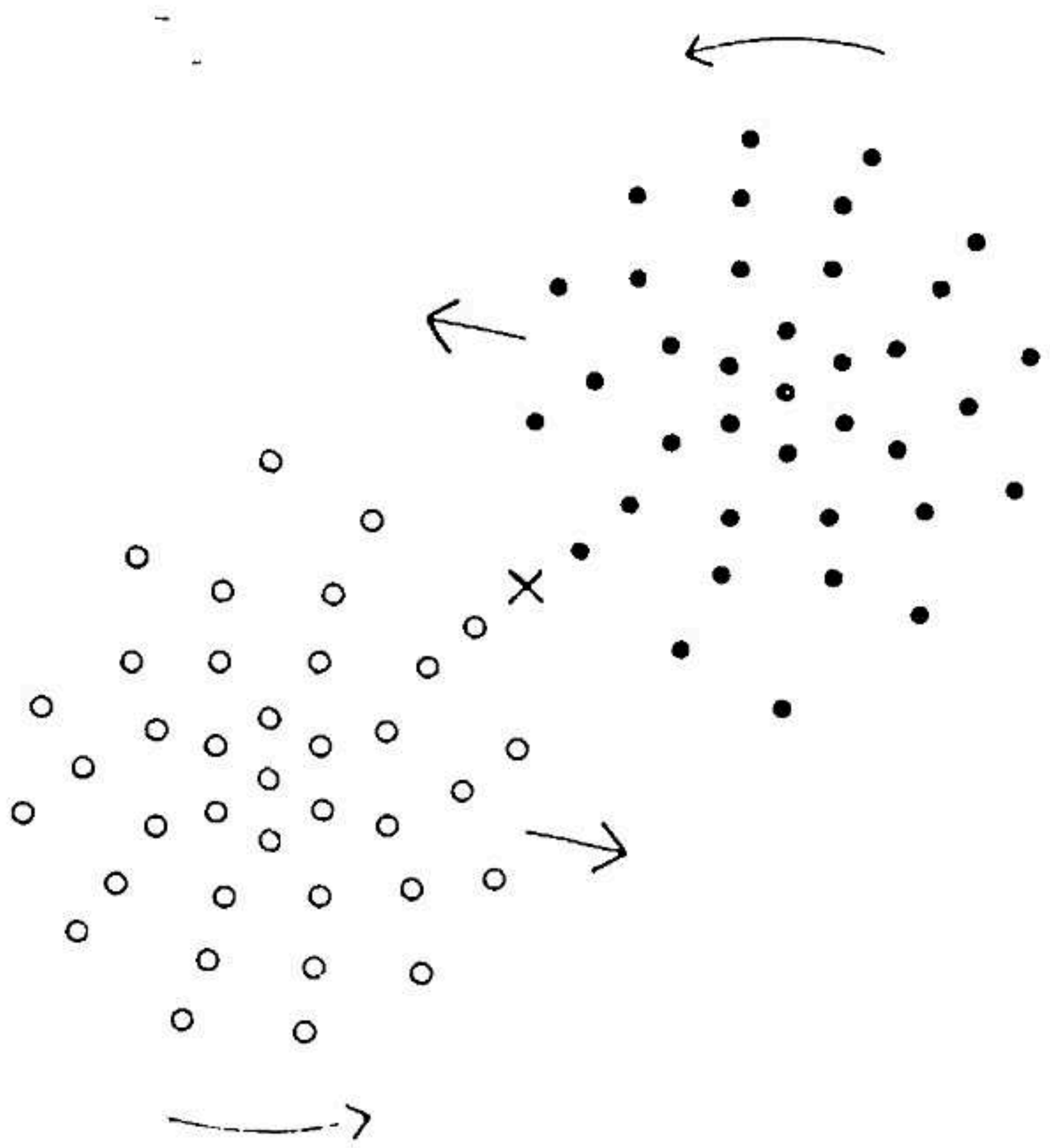
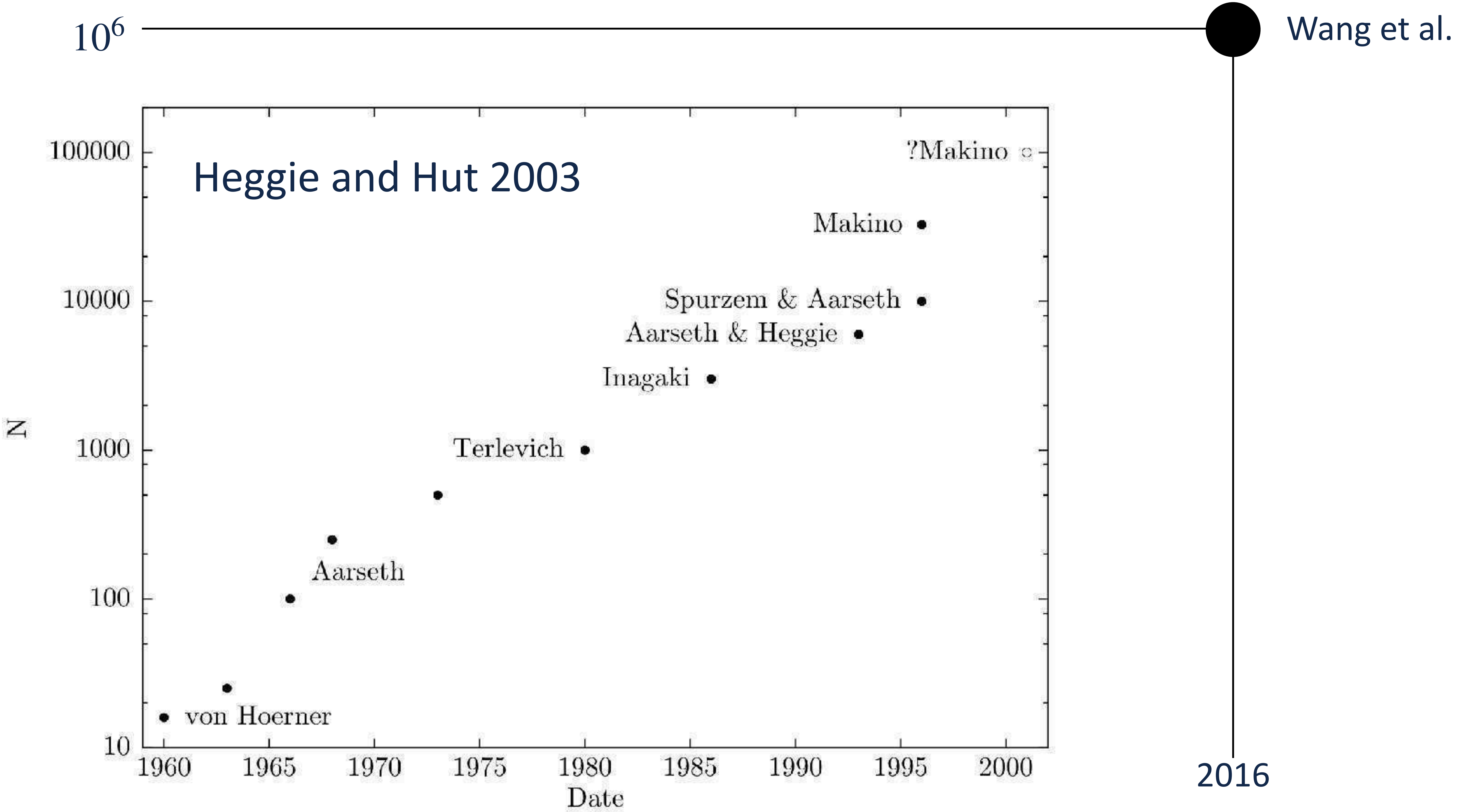


Gravitational *N*-body Problem



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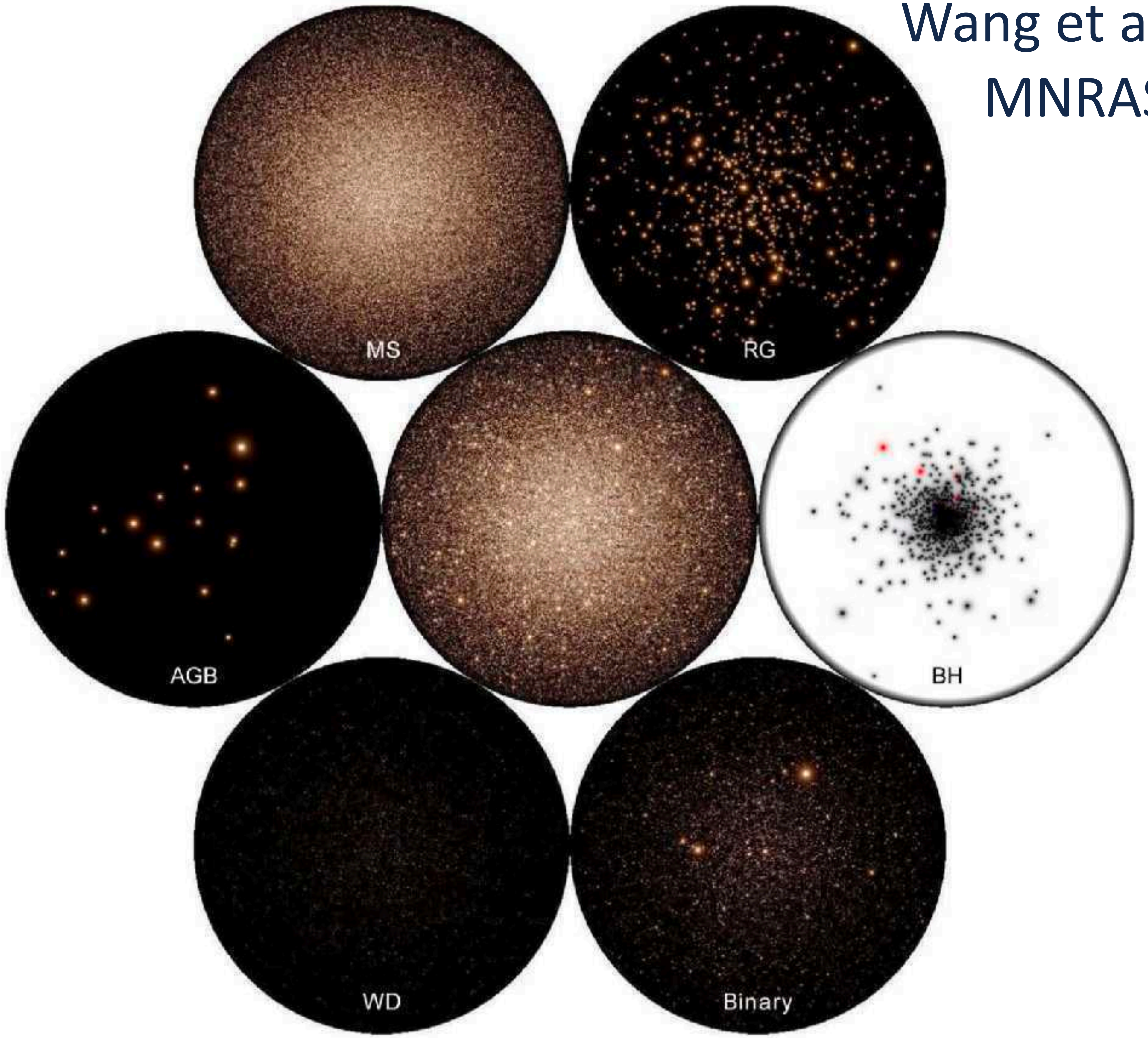

Wang et al. (2016)
MNRAS, **458**, 2

Still took ~ 13 **months** of wall time!

Gravitational *N*-body Problem

Core issues:

1. Close encounters are the whole point!
2. Wide range of timescales
3. Scales as N^2

GRAPE (Gravity Pipe)

Ways to Optimize

1. Hardware optimization
2. Algorithmic enhancements
   - e.g. **FMM**, which scales as $\mathcal{O}(N)$
   - Mukherjee el al.+CLR (2021,2023)
3. Simplifying assumptions via physics

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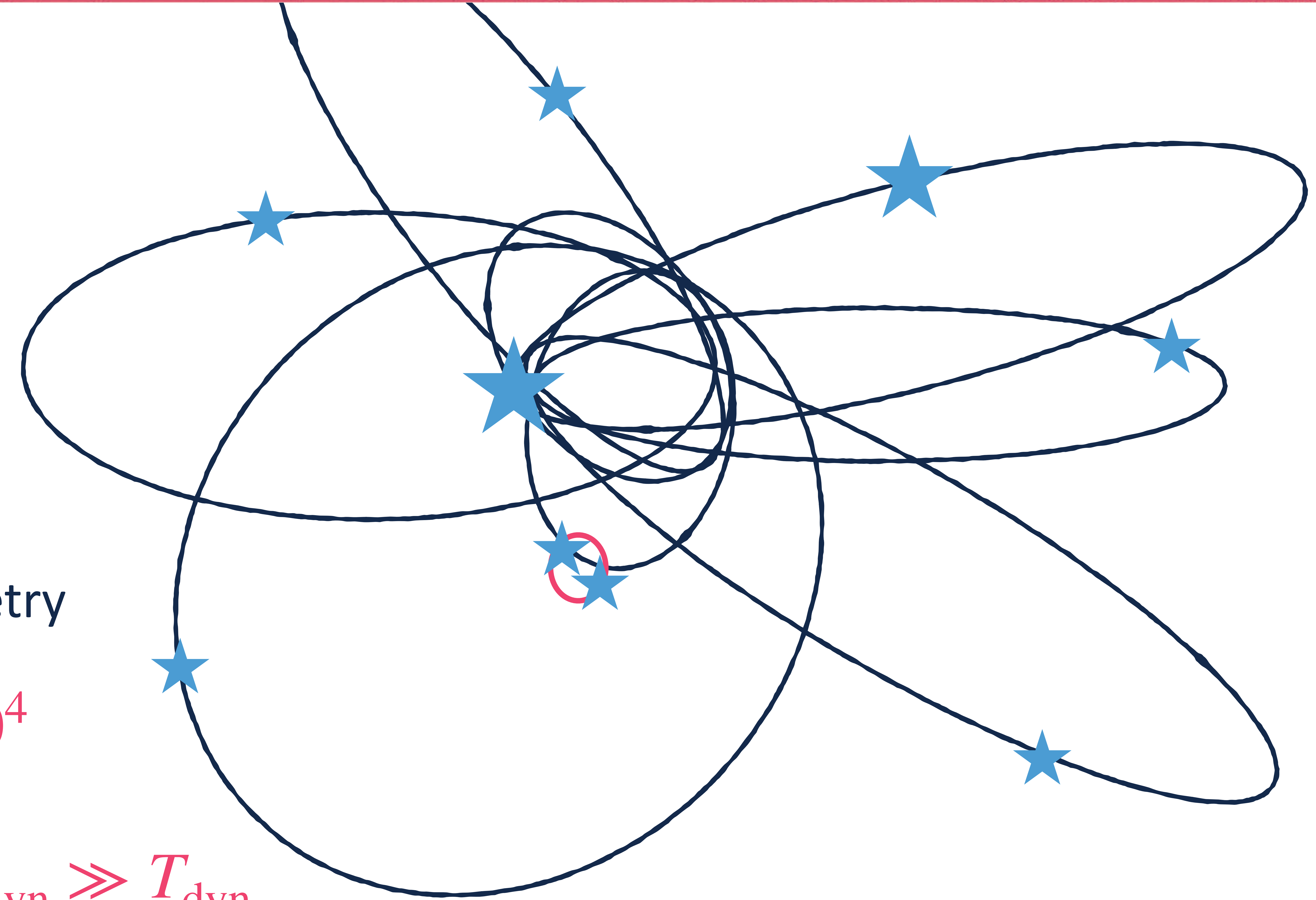
Monte Carlo *N*-body Problem

Three Core Assumptions:

1. Boltzmann's Molecular Chaos Assumption
2. Assume large N (Fokker-Planck approx.)
3. Spherical Symmetry



$$N \gtrsim 10^4$$

$$T_{\rm relax} \approx \frac{0.1N}{\log N} T_{\rm dyn} \gg T_{\rm dyn}$$

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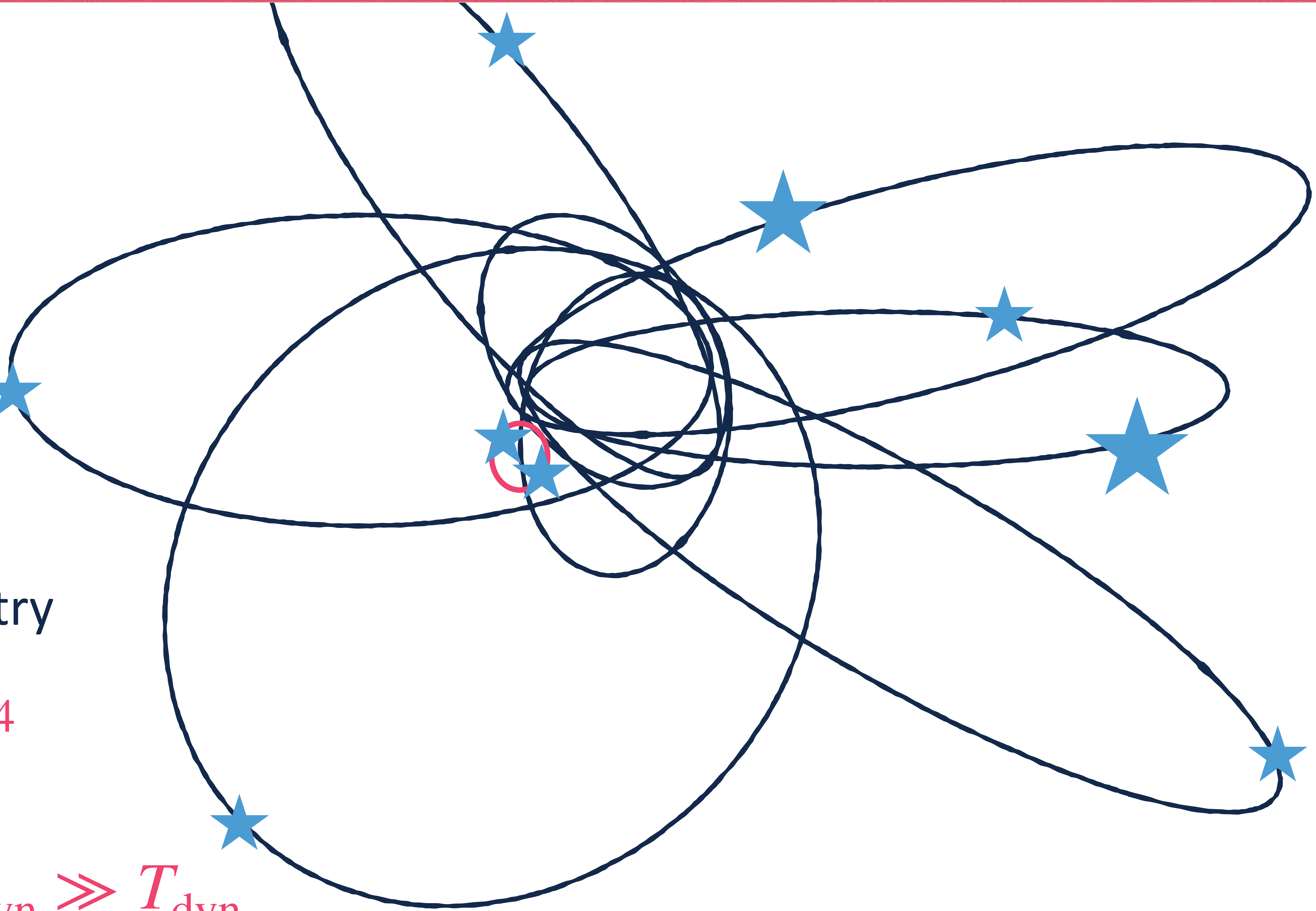

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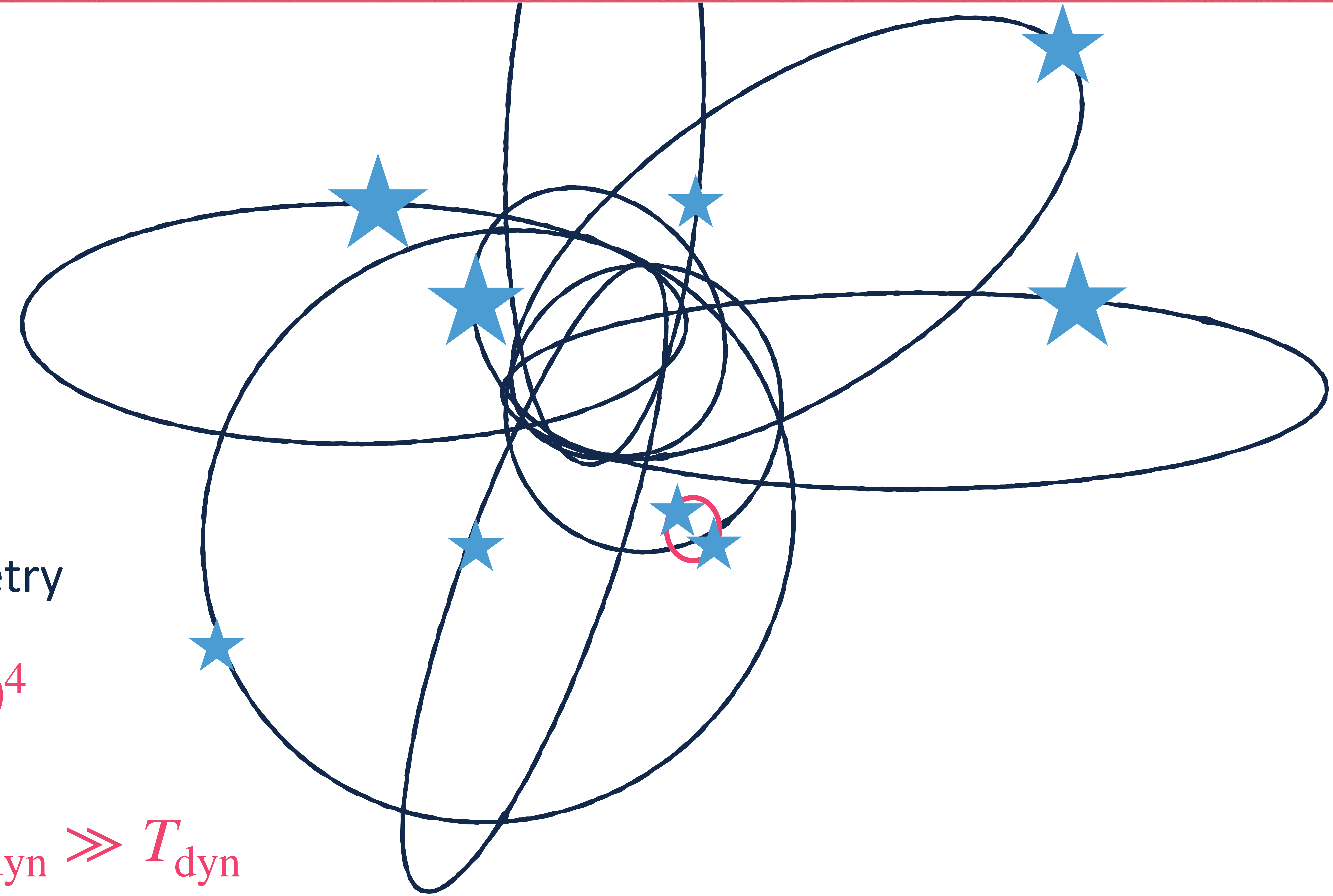

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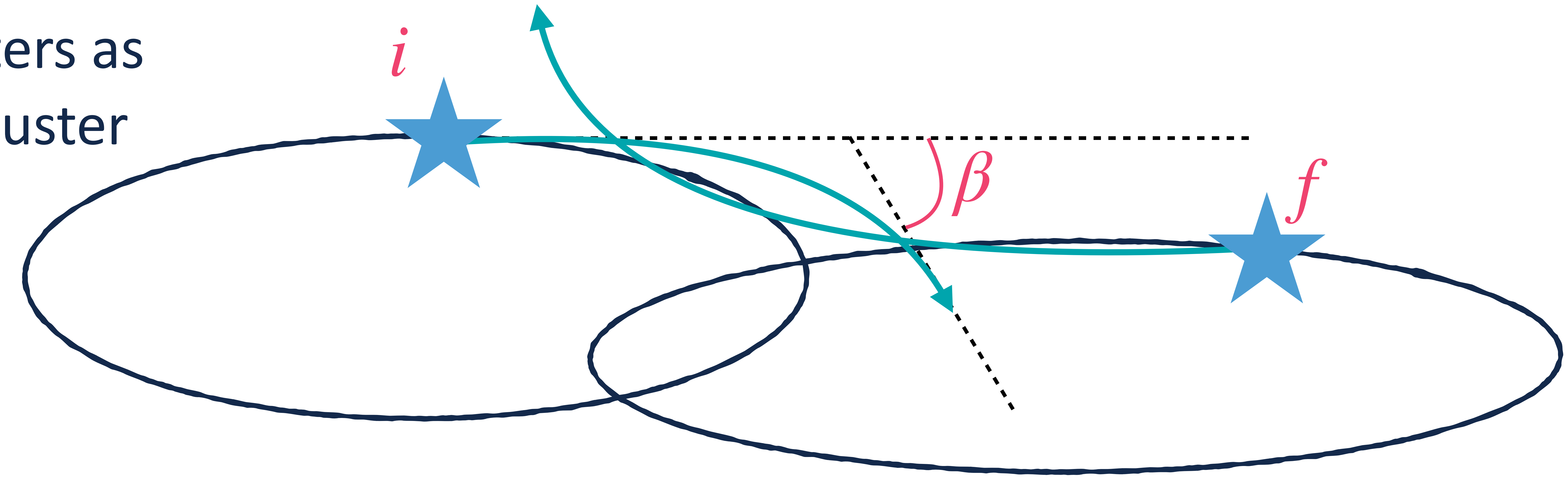
Monte Carlo *N*-body Problem

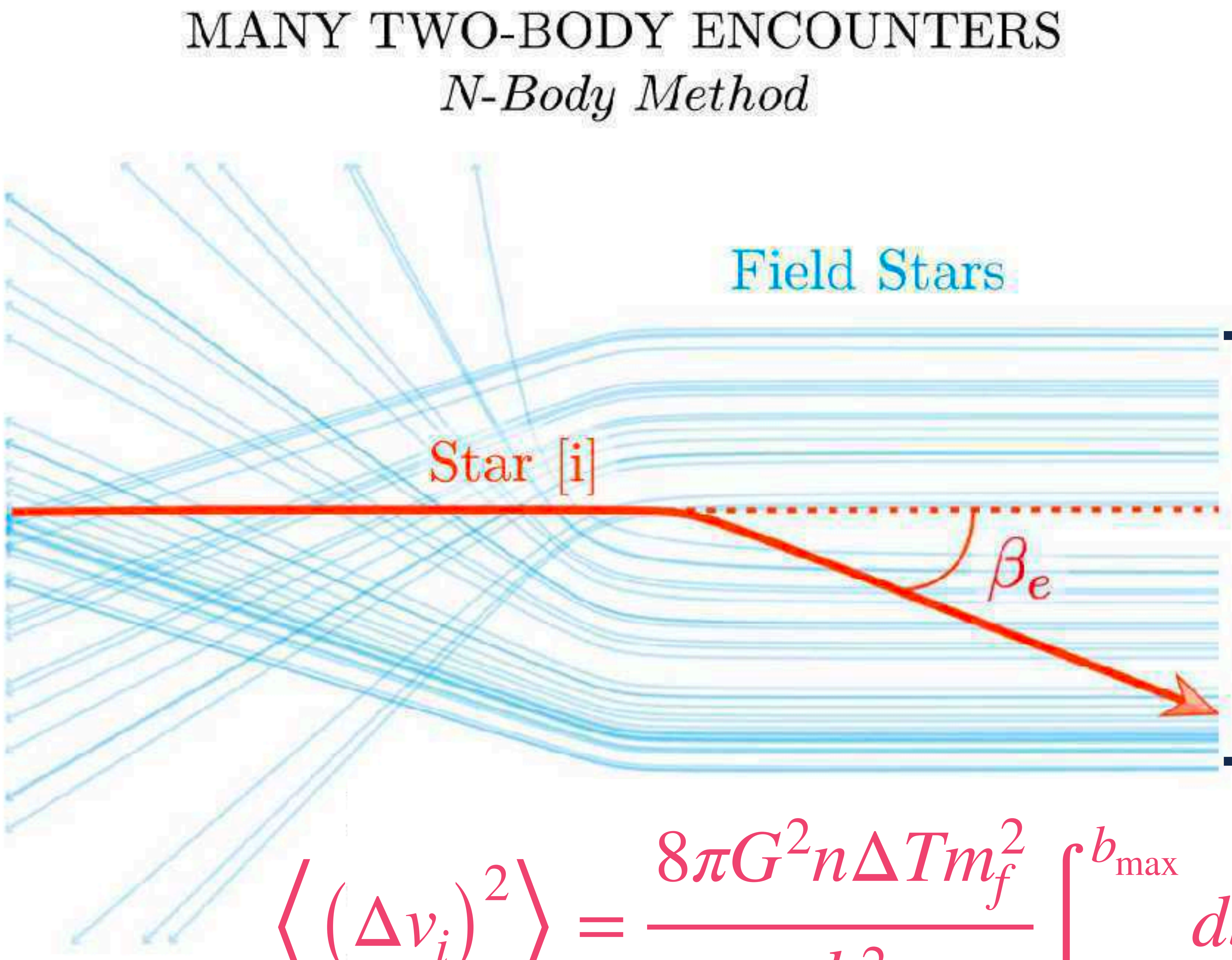
Particles experience two-body encounters as they orbit in the cluster



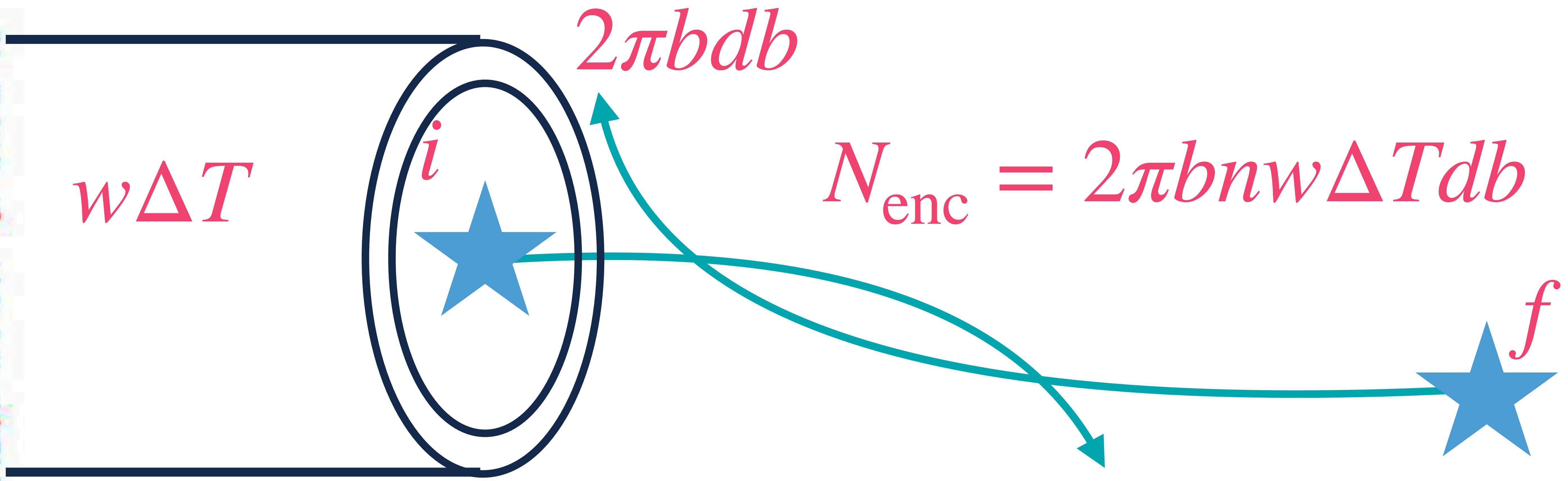
$$(\Delta v_i)^2 = \frac{4m_f^2}{(m_i + m_f)} w^2 \sin^2\left(\frac{\beta}{2}\right)$$ where $w = |\vec{v}_i - \vec{v}_f|$

$$= \frac{4G^2 m_f^2}{w^2 b_{90}^2} \frac{1}{1 + (b/b_{90})^2}$$ where $b_{90} \equiv \frac{G(m_i + m_f)}{w^2}$ is the impact parameter where $\beta = 90°$

Monte Carlo *N*-body Problem



$$(\Delta v_i)^2 = \frac{4G^2 m_f^2}{w^2 b_{90}^2} \frac{1}{1 + (b/b_{90})^2}$$



$$N_{\mathrm{enc}} = 2\pi b n w \Delta T db$$

$$\left\langle (\Delta v_i)^2 \right\rangle = \frac{8\pi G^2 n \Delta T m_f^2}{w b_{90}^2} \int_0^{b_{\max}} db \frac{b}{1 + (b/b_{90})^2}$$

$$= 4\pi G^2 n \Delta T m_f^2 w^{-1} \log\left(1 + (b_{\max}/b_{90})^2\right) \simeq 8\pi G^2 n \Delta T m_f^2 w^{-1} \log \Lambda$$

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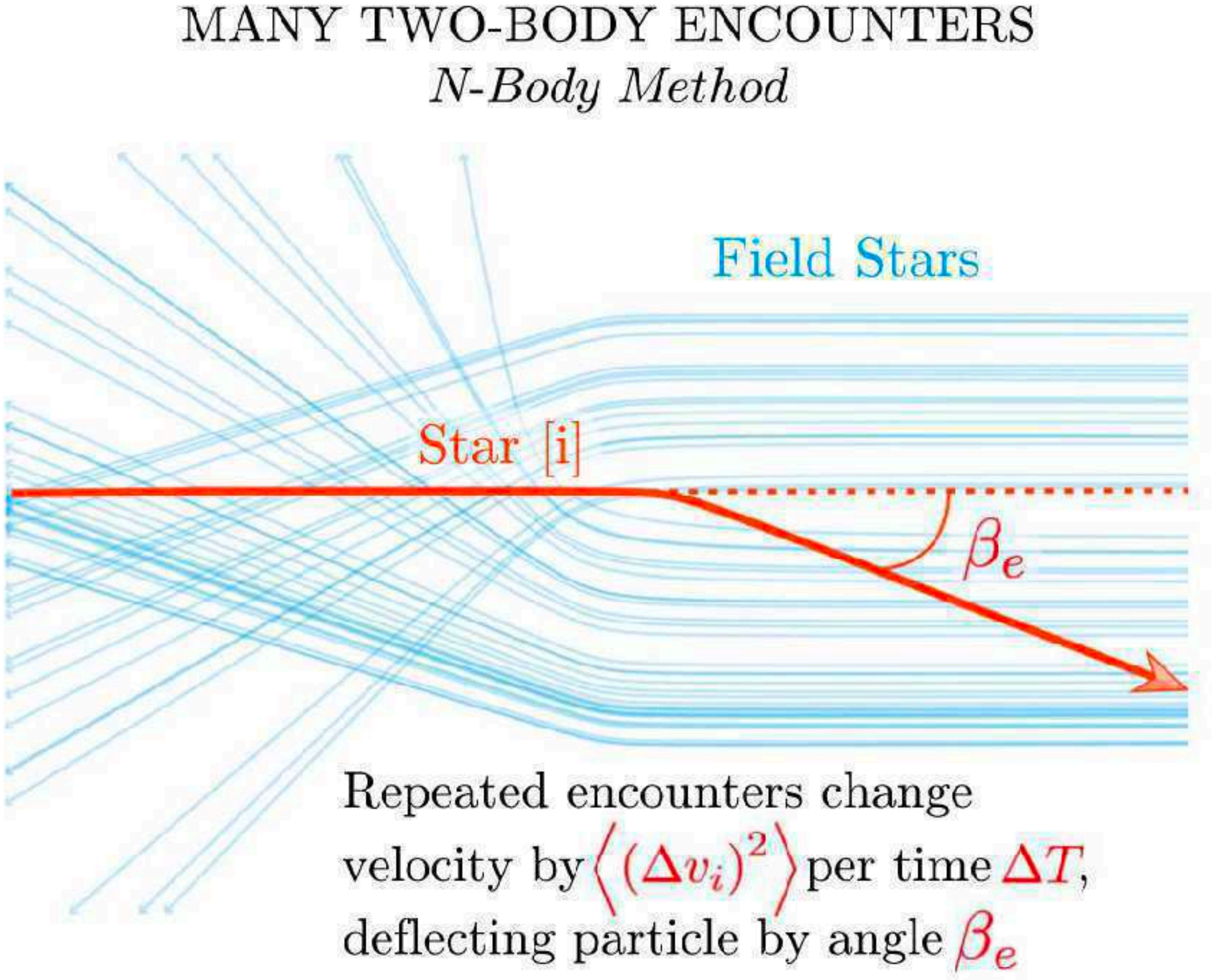
Monte Carlo *N*-body Problem



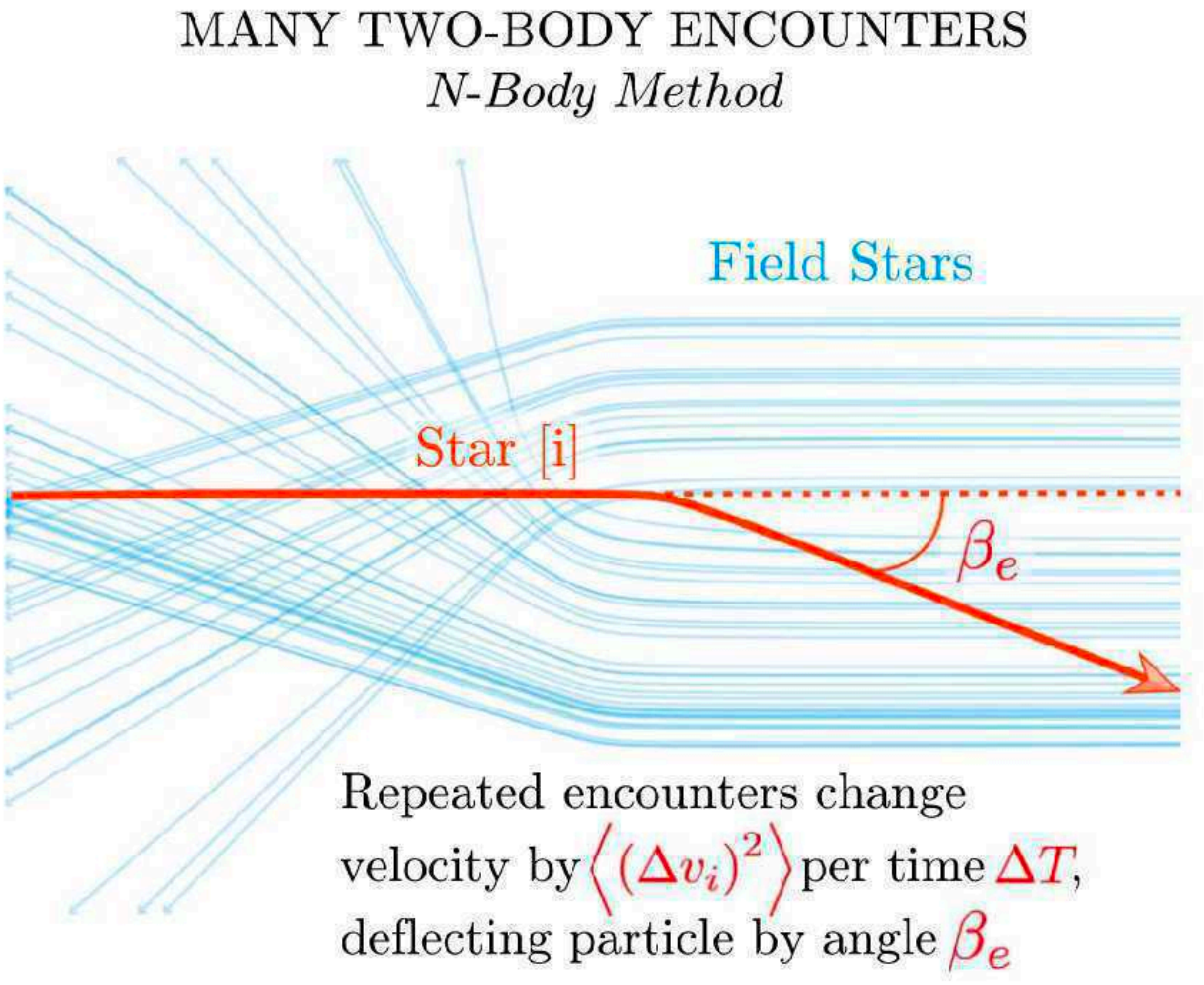
In reality each encounter should have a unique mass and relative velocity drawn from the distribution function

$$\left\langle (\Delta v_i)^2 \right\rangle \simeq 8\pi G^2 n \Delta T \langle m_f^2 w^{-1} \rangle_F \log \Lambda$$

where $\langle m_f^2 w^{-1} \rangle_F \equiv \int F_i F_f m_f^2 w^{-1} d^3\vec{v}_i d^3\vec{v}_f dm_i dm_f$

For $F_i = F(\vec{r}_i, \vec{v}_i, m_i)$, $F_f = F(\vec{r}_f, \vec{v}_f, m_f)$

Monte Carlo *N*-body Problem



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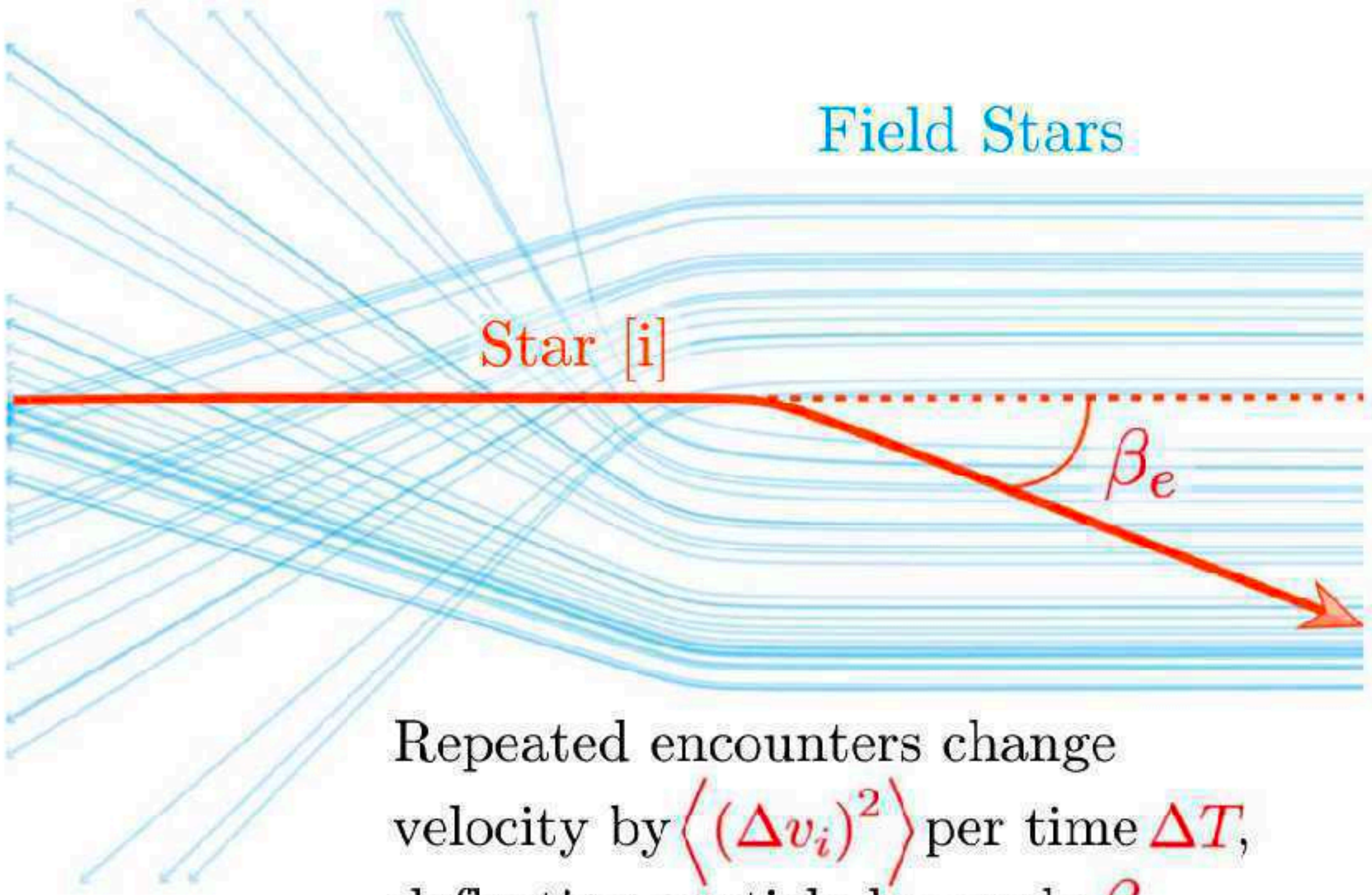
Monte Carlo *N*-body Problem

MANY TWO-BODY ENCOUNTERS

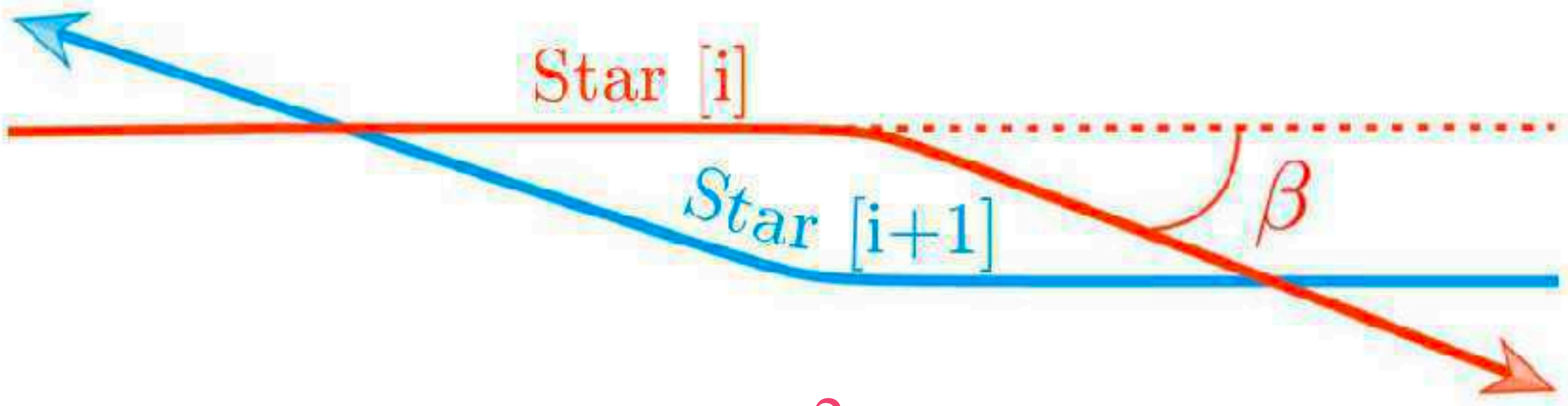
N-Body Method



Repeated encounters change velocity by $\left\langle (\Delta v_i)^2 \right\rangle$ per time ΔT, deflecting particle by angle β_e

EFFECTIVE TWO-BODY RELAXATION

Hénon's Method



$$(\Delta v_i)^2 = \frac{4m_f^2}{(m_i + m_f)^2} w^2 \sin^2\left(\frac{\beta}{2}\right)$$

$$8\pi G^2 n \Delta T \langle m_f^2 w^{-1} \rangle_F \log\Lambda = \left\langle \frac{4m_f^2 w^2}{(m_i + m_f)^2} \right\rangle_F \sin^2\left(\frac{\beta_e}{2}\right)$$

$$\left\langle (\Delta v_i)^2 \right\rangle \simeq 8\pi G^2 n \Delta T \langle m_f^2 w^{-1} \rangle_F \log\Lambda$$

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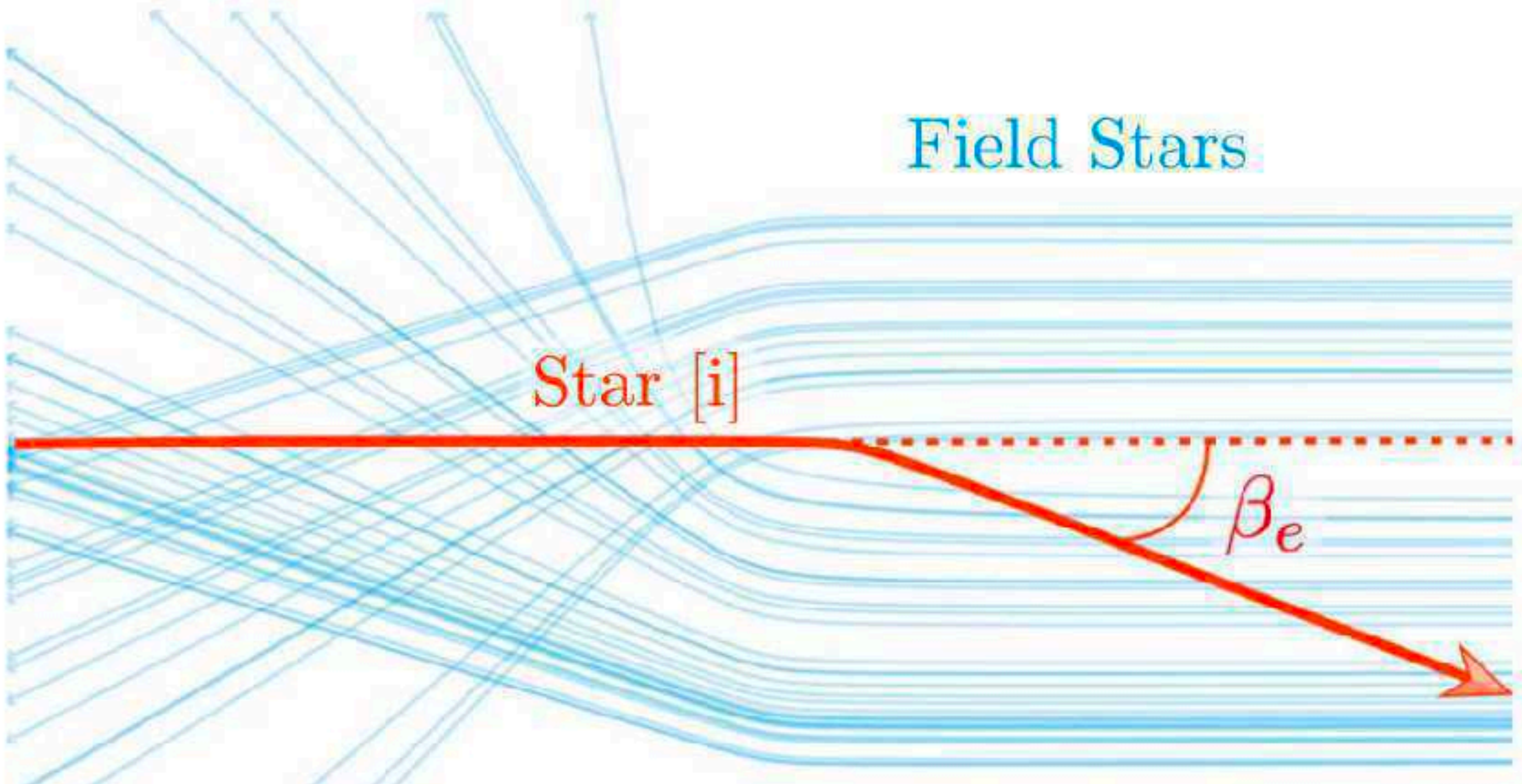
Monte Carlo *N*-body Problem

MANY TWO-BODY ENCOUNTERS

N-Body Method

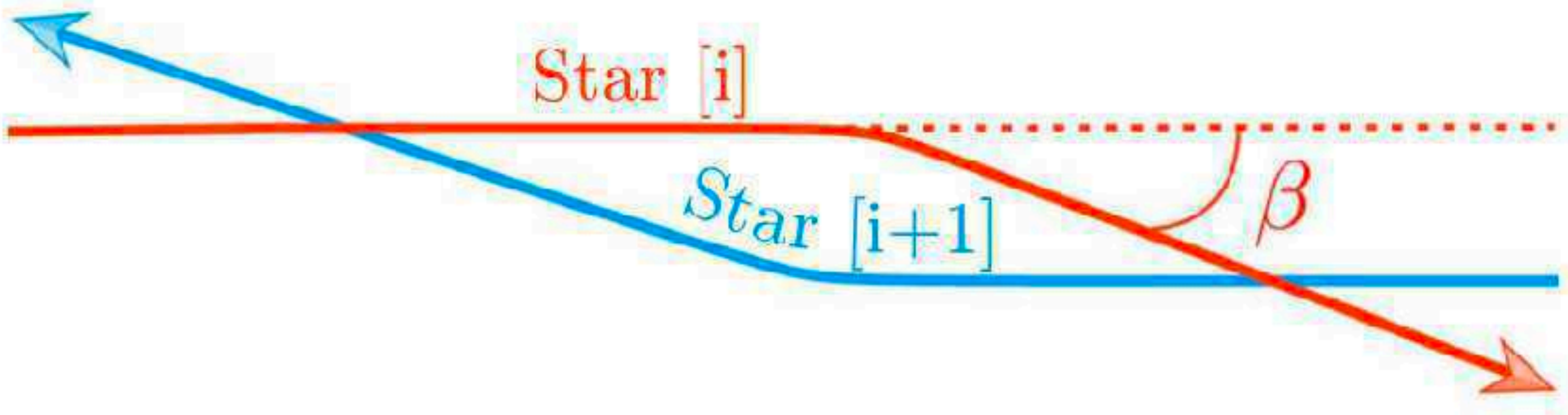


Repeated encounters change velocity by $\left\langle (\Delta v_i)^2 \right\rangle$ per time ΔT, deflecting particle by angle β_e

EFFECTIVE TWO-BODY RELAXATION

Hénon's Method

Perform "effective" scatterings between nearest neighbors, setting $\beta \equiv \beta_e$



Reproduces statistics of many two-body encounters, changing particle's velocity by $(\Delta v_i)^2 \propto \sin^2(\beta/2)$ every timestep ΔT

How do we calculate $\langle \cdots \rangle_F$?

Assume our nearest star $(i+1)$ is a fair draw from $F(\vec{r}, \vec{v}, m)$

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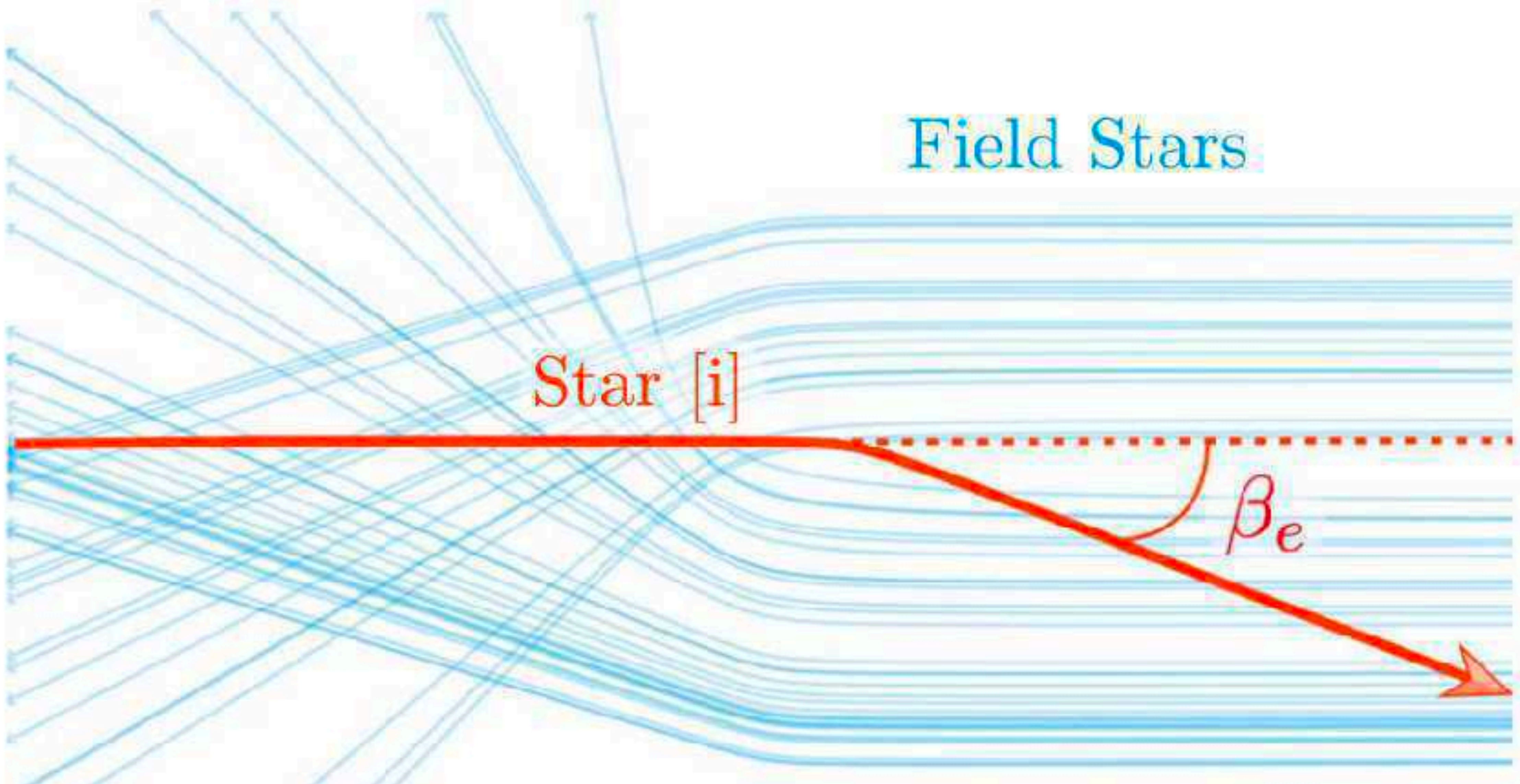

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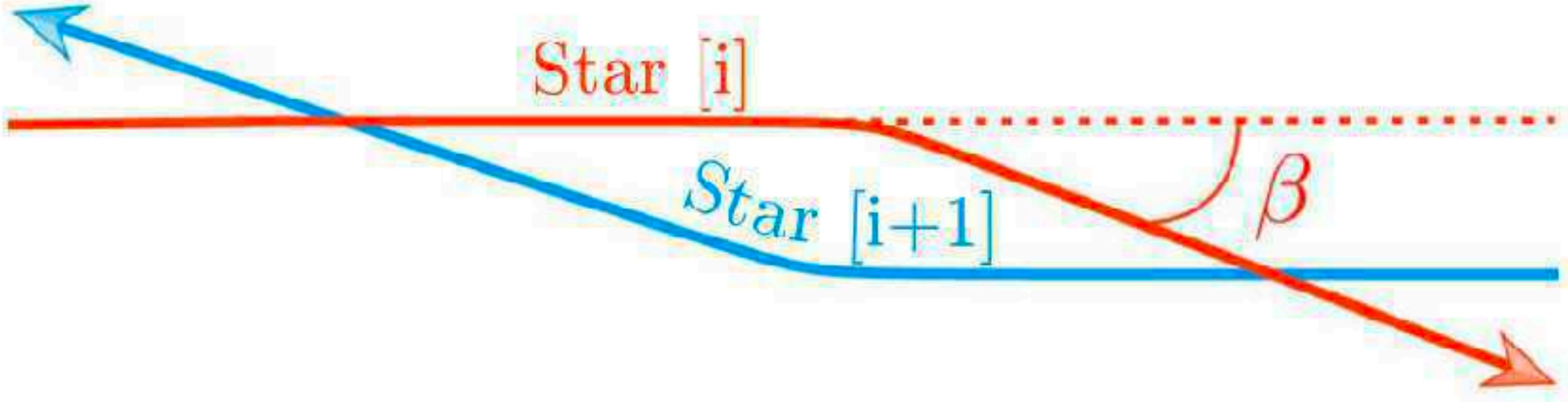

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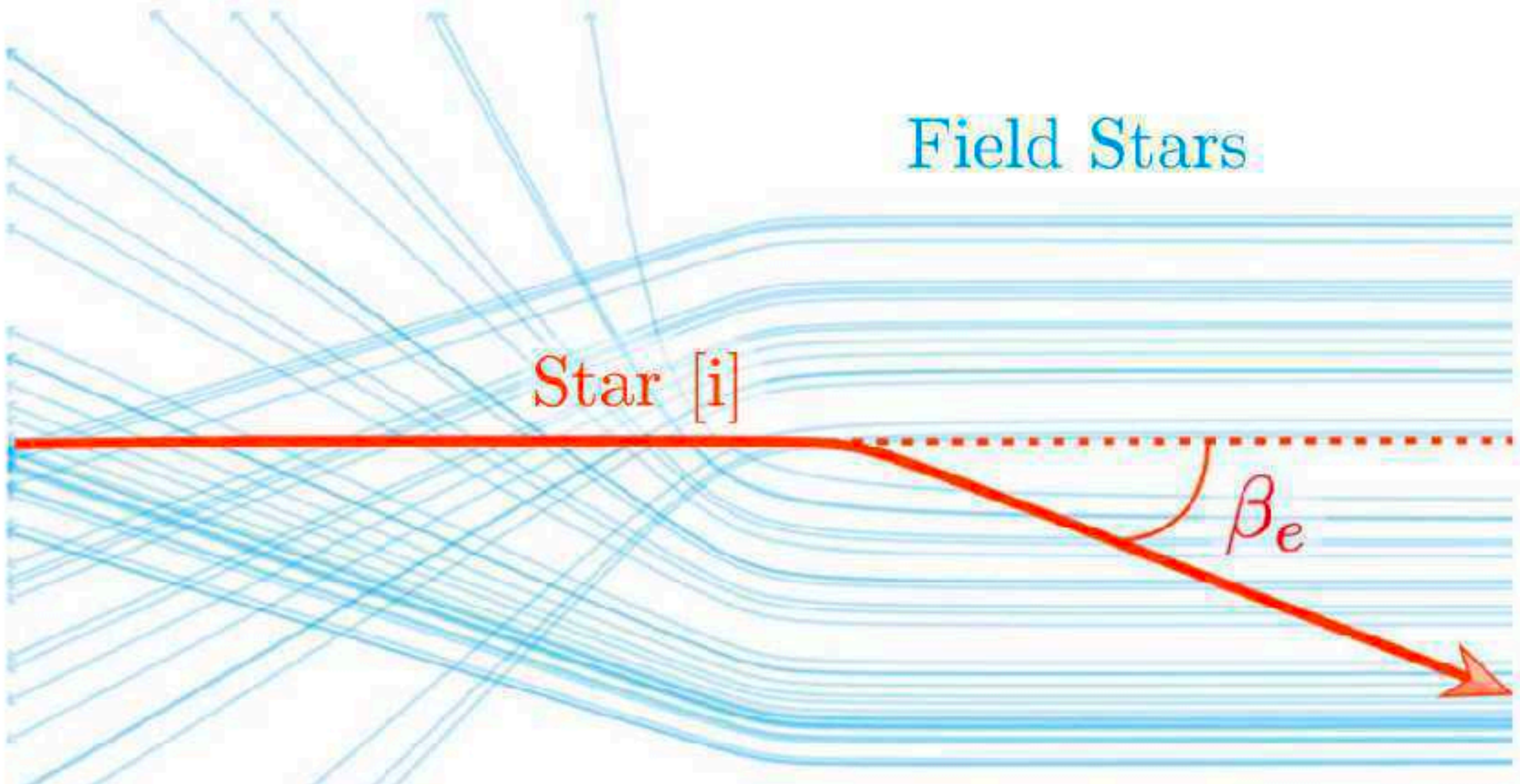

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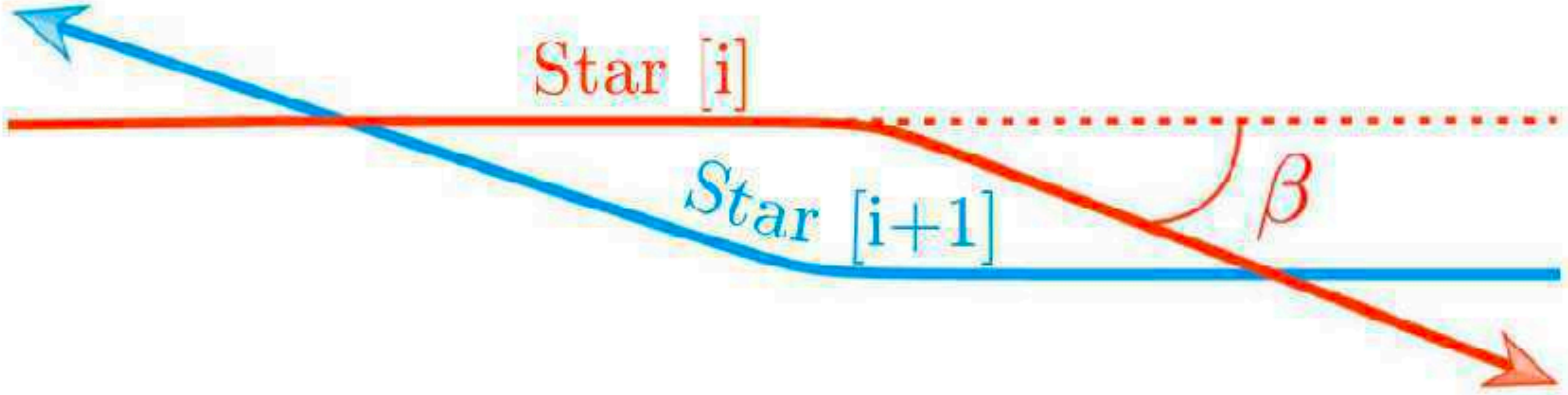

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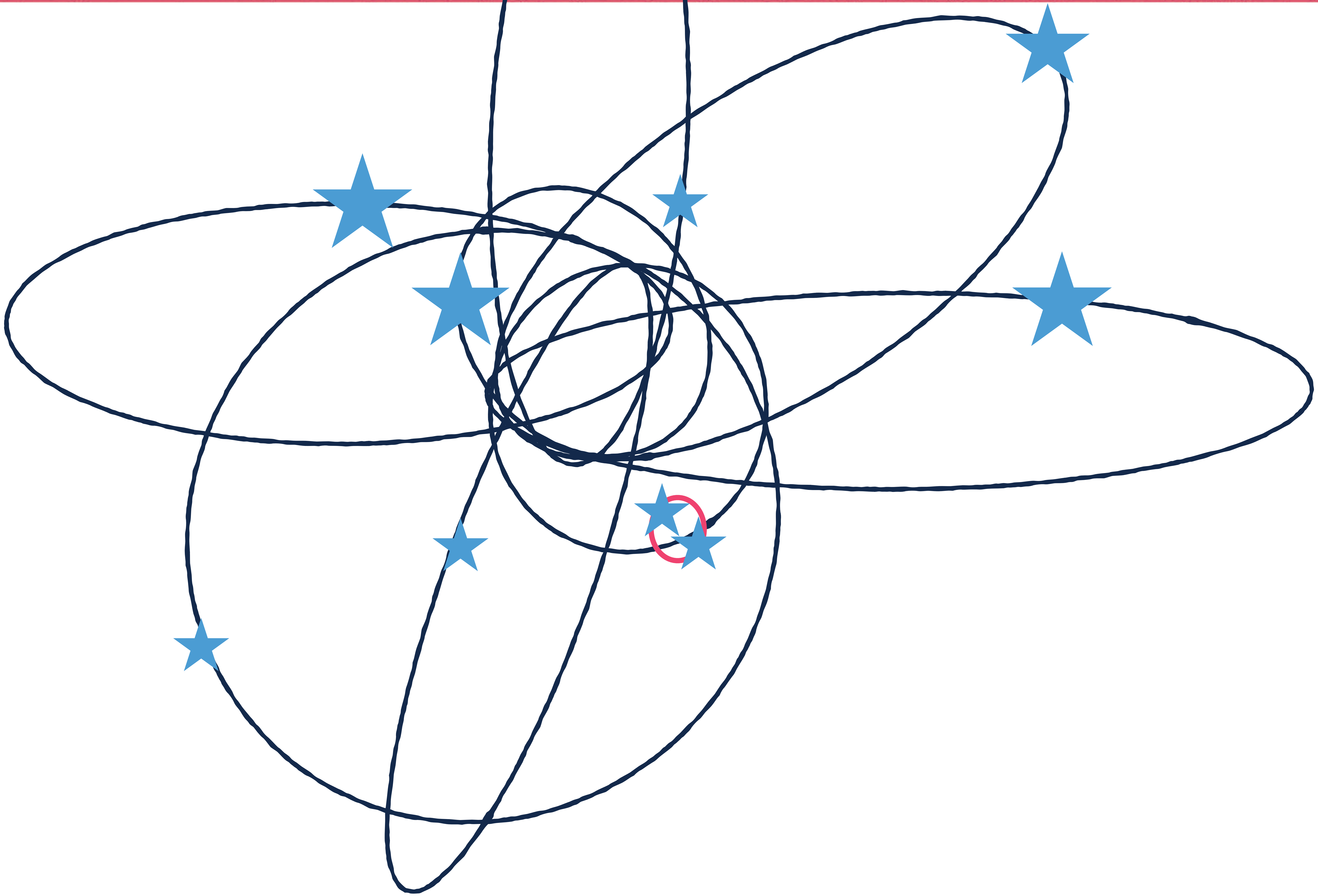
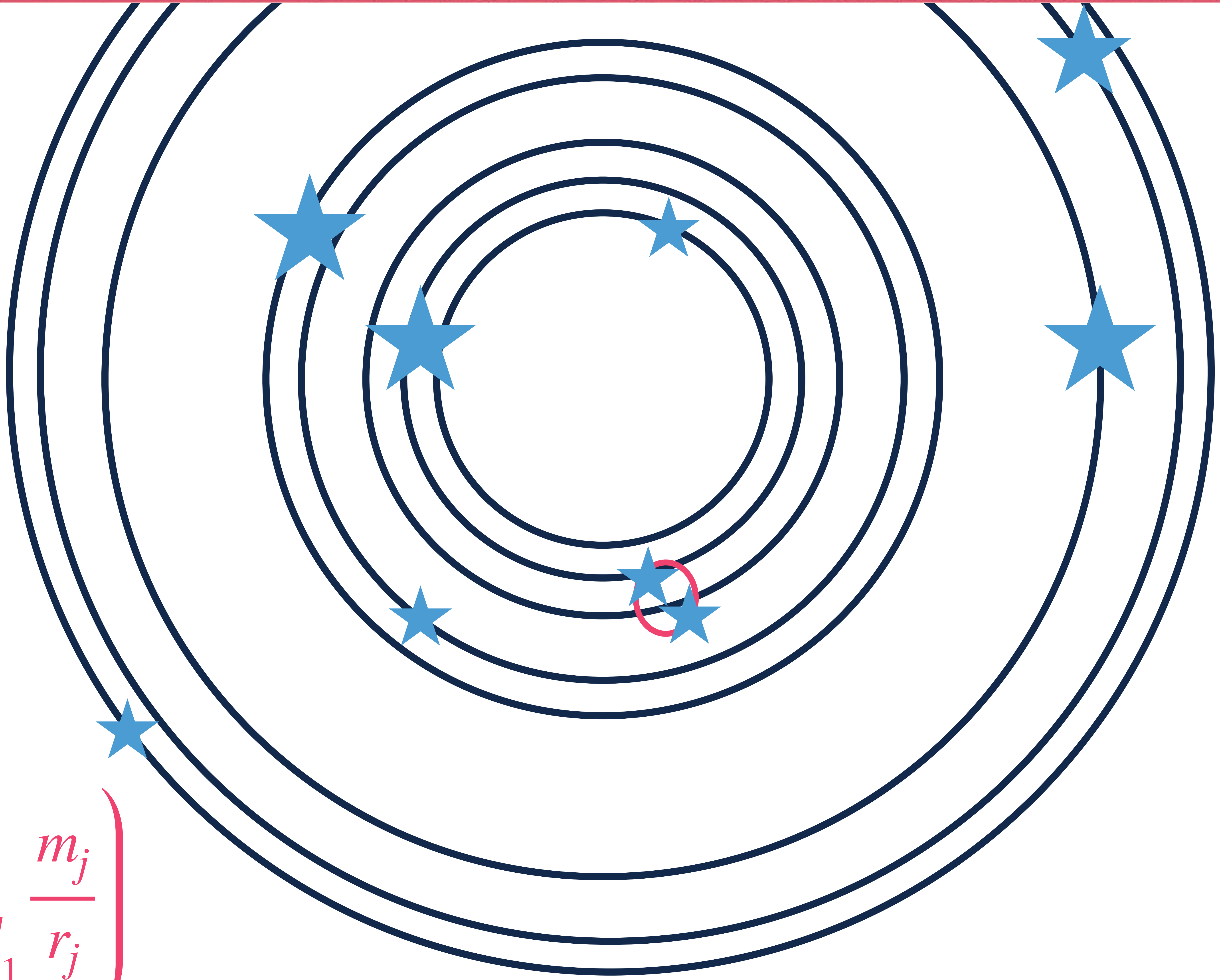
Hénon's method in practice



Hénon's method in practice

1. Assume **spherical symmetry**, and that stars can be represented as **spherical mass shells**



Potential is trivial;
for $r_i < r < r_{i+1}$

$$\Phi(r) = -G\left(\frac{M_i}{r} + \sum_{j=i+1}^{N} \frac{m_j}{r_j}\right)$$

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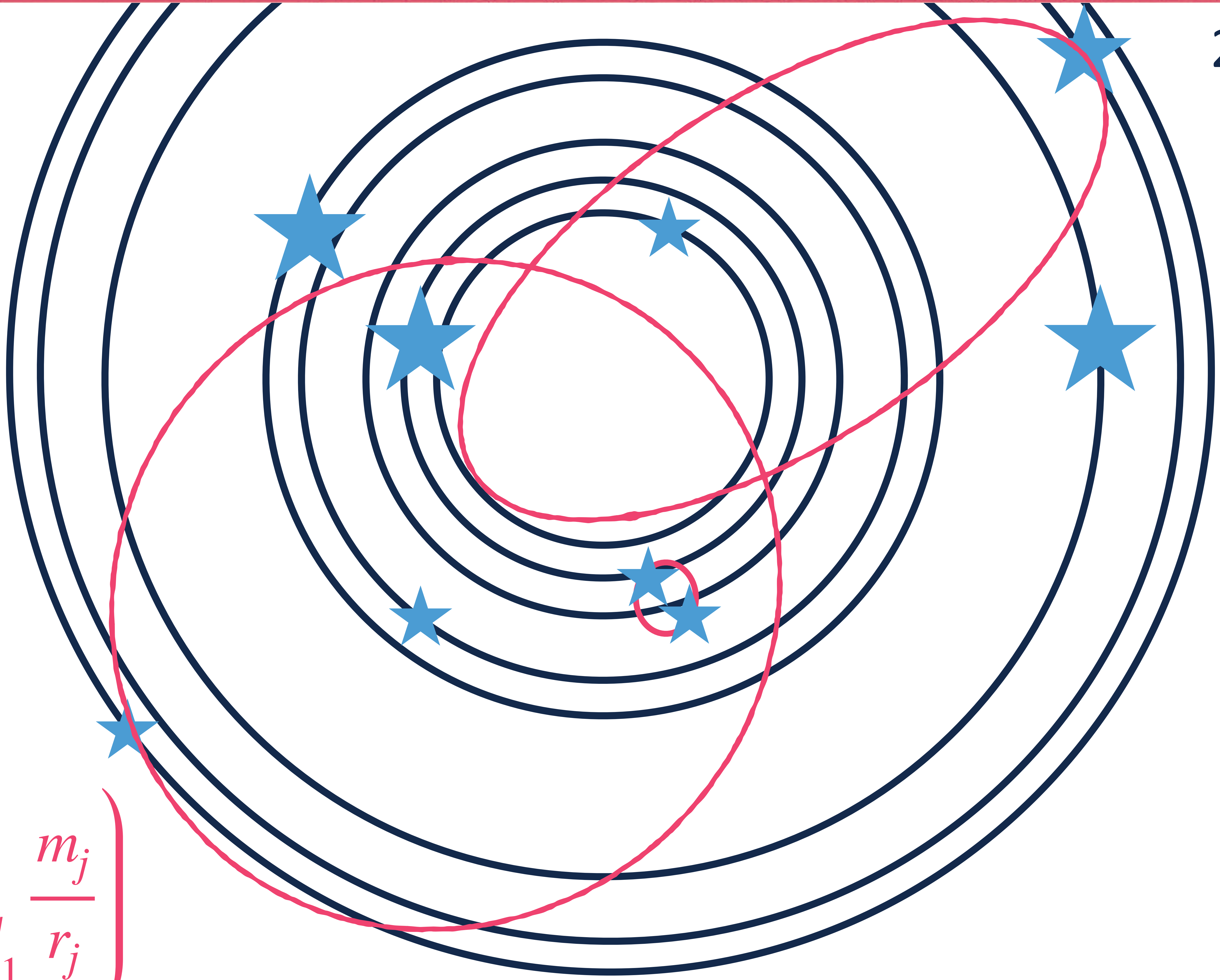

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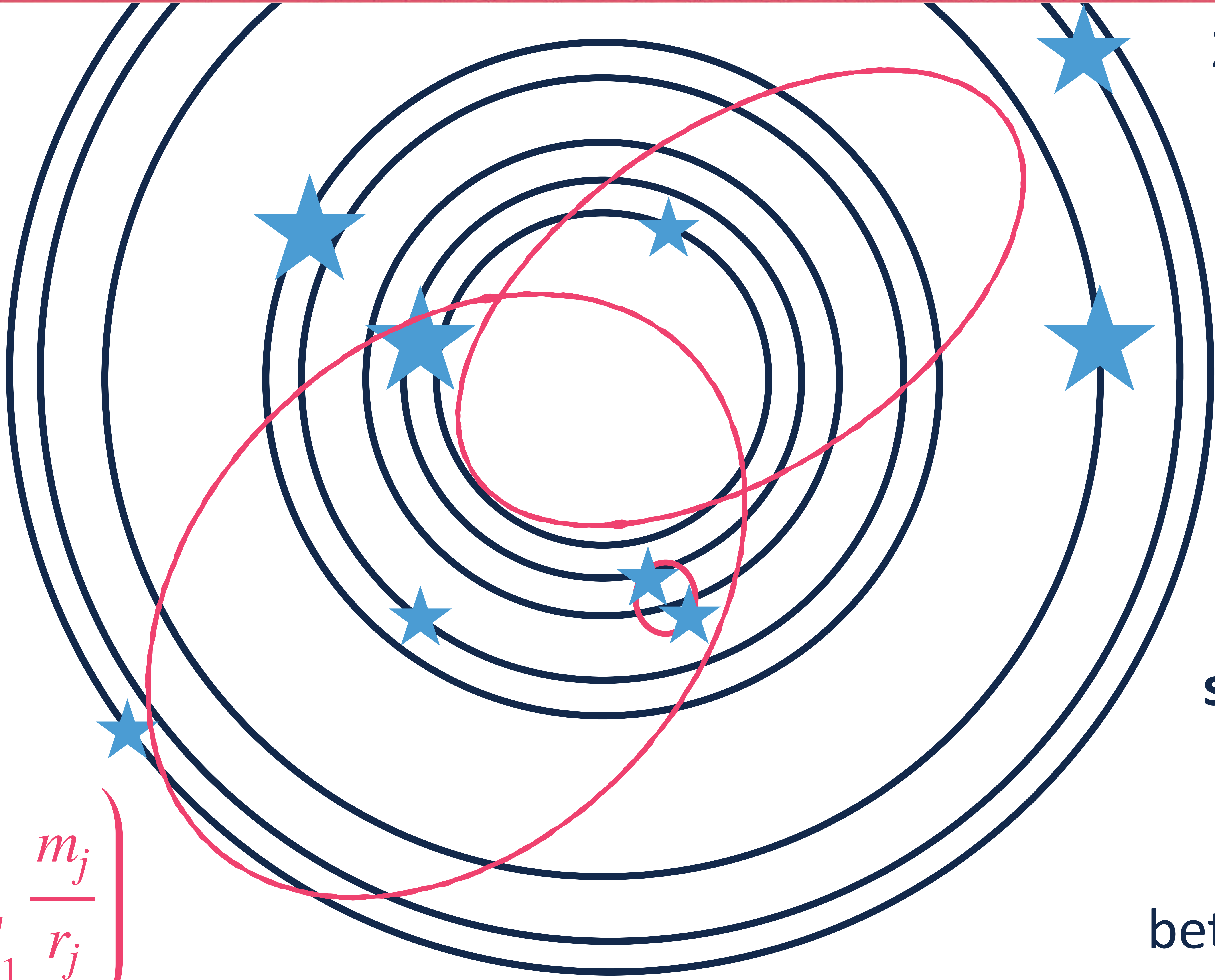

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Can also consider **strong encounters**, computing

$$P_{\rm strong} = nw\sigma\Delta T$$

between neighboring stars and binaries

Hénon's method in practice

3. Get **new positions** in spherical cluster potential

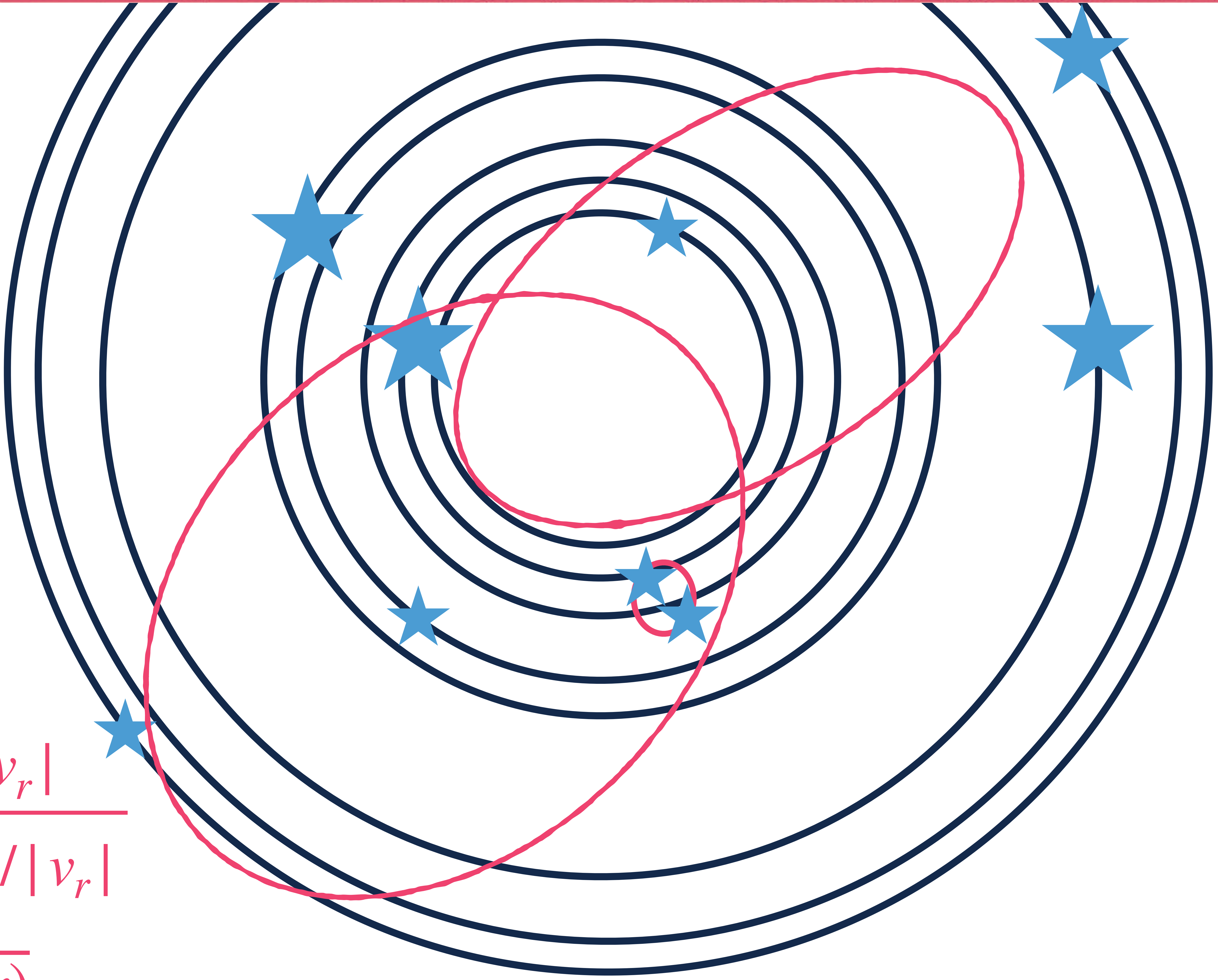
Energy equation:

$$Q(r) = 2E - 2\Phi(r) - J^2/r^2$$

Then just sample orbits from radius distribution

$$P(r)dr = \frac{dt}{T} = \frac{dr/|v_r|}{\int_{r_{\min}}^{r_{\max}} dr/|v_r|}$$

Where $v_r = \sqrt{Q(r)}$



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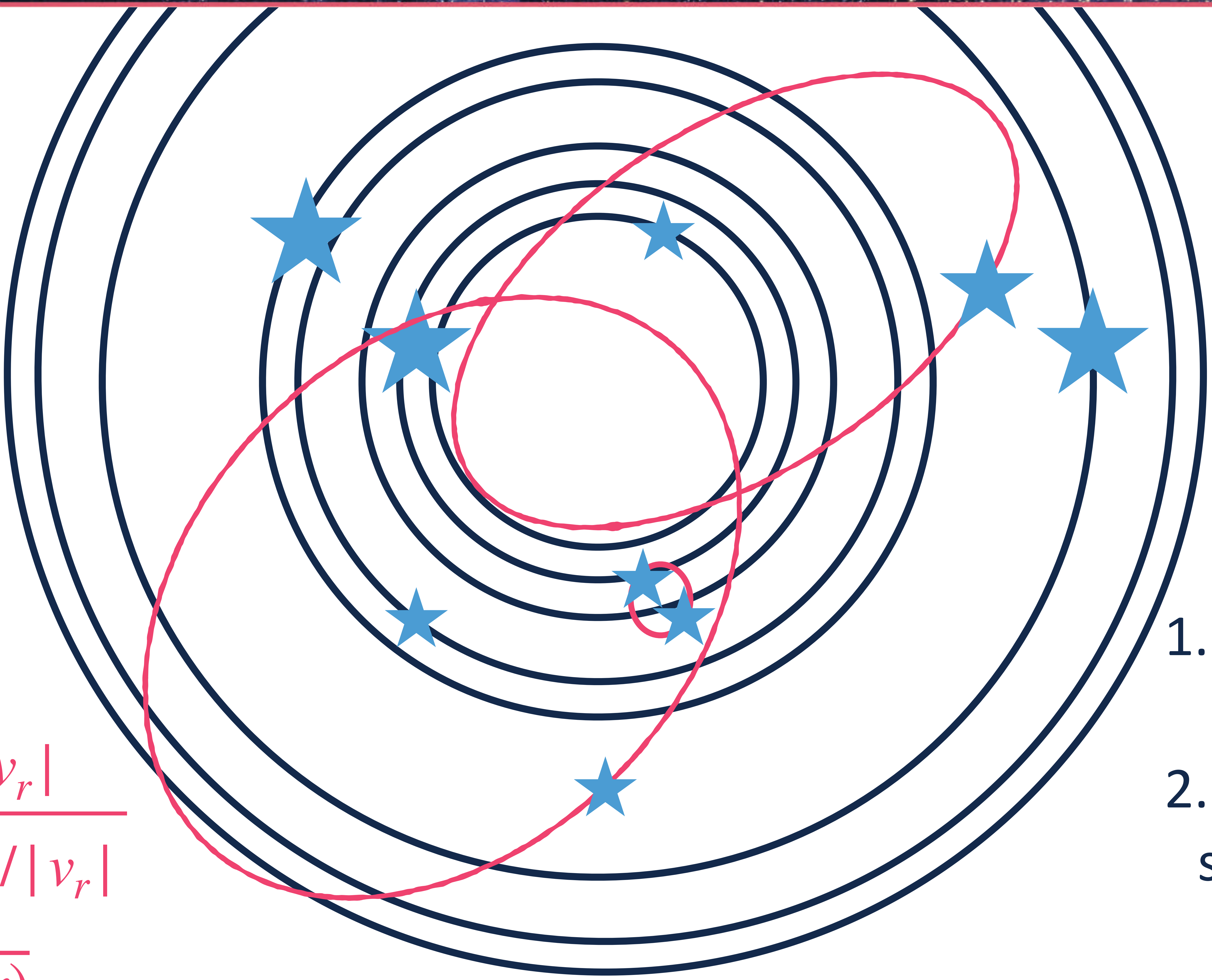


4. **Sort** particles by increasing radius, recompute potential, back to step 1

Two points:

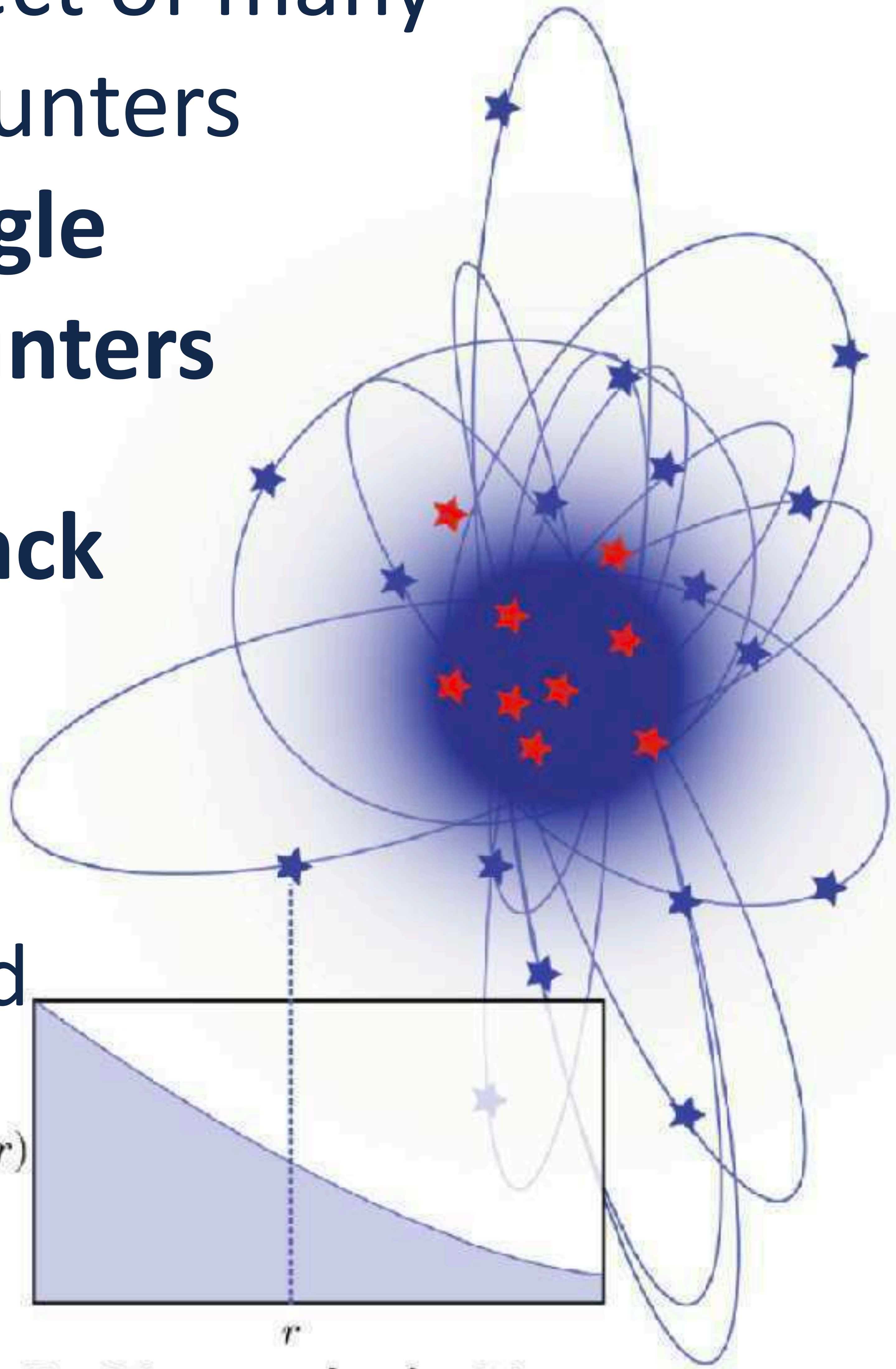
1. Sorting is $N \log N$ complexity
2. Monte Carlo sampling is used in **two different places**

Cluster Monte Carlo Code, CMC

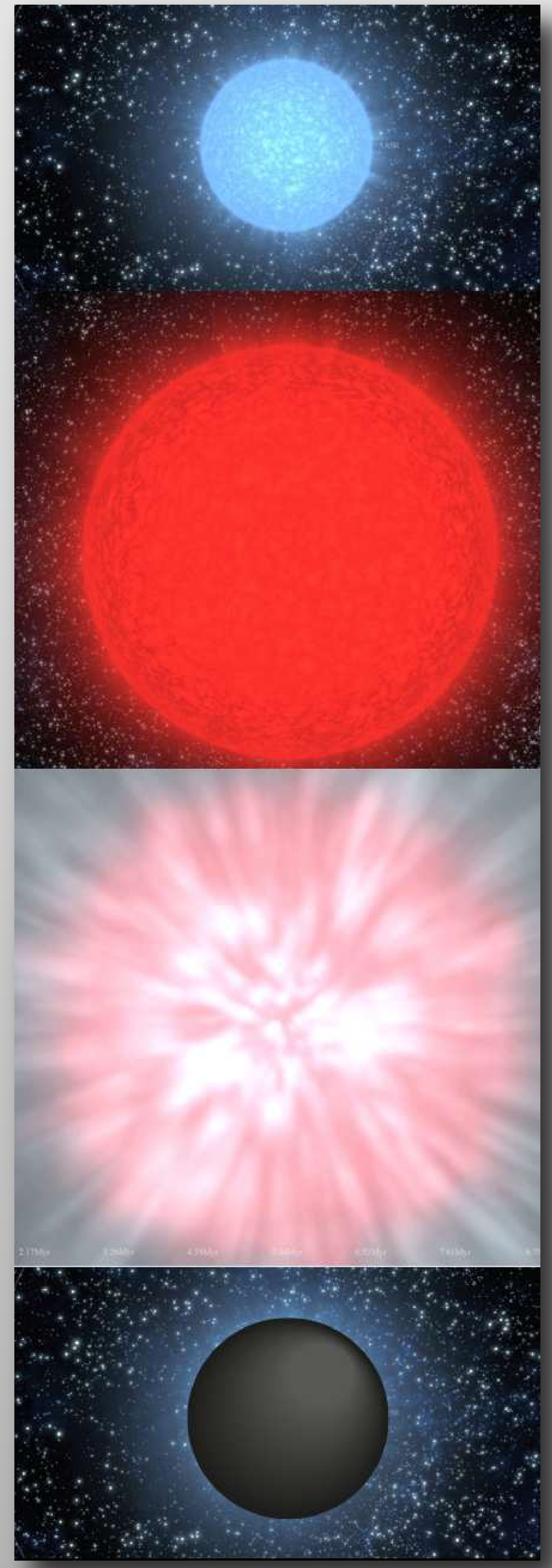
Cumulative effect of many two-body encounters modeled as **single effective encounters**

(i.e. **Fokker-Planck Approximation**)

Positions and velocities sampled from orbits in spherical potential



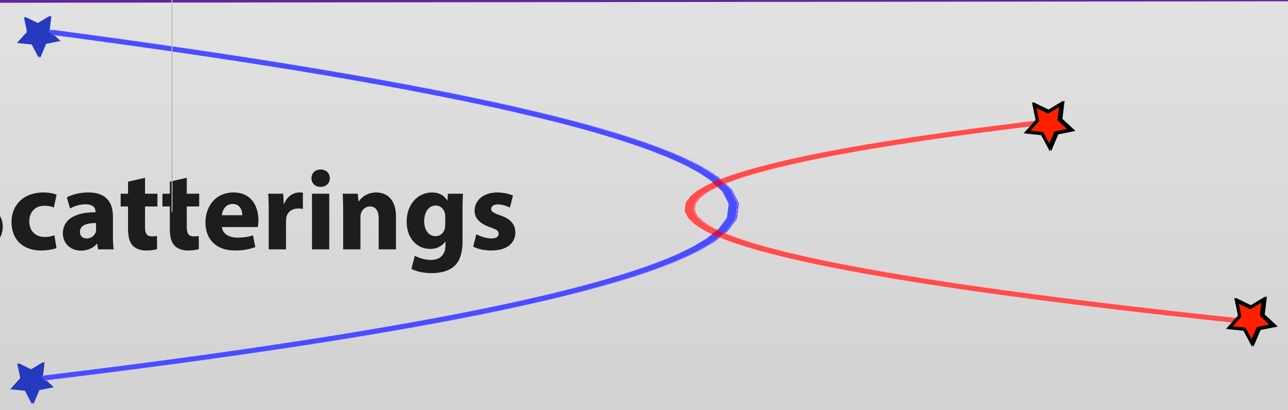
Stellar Evolution



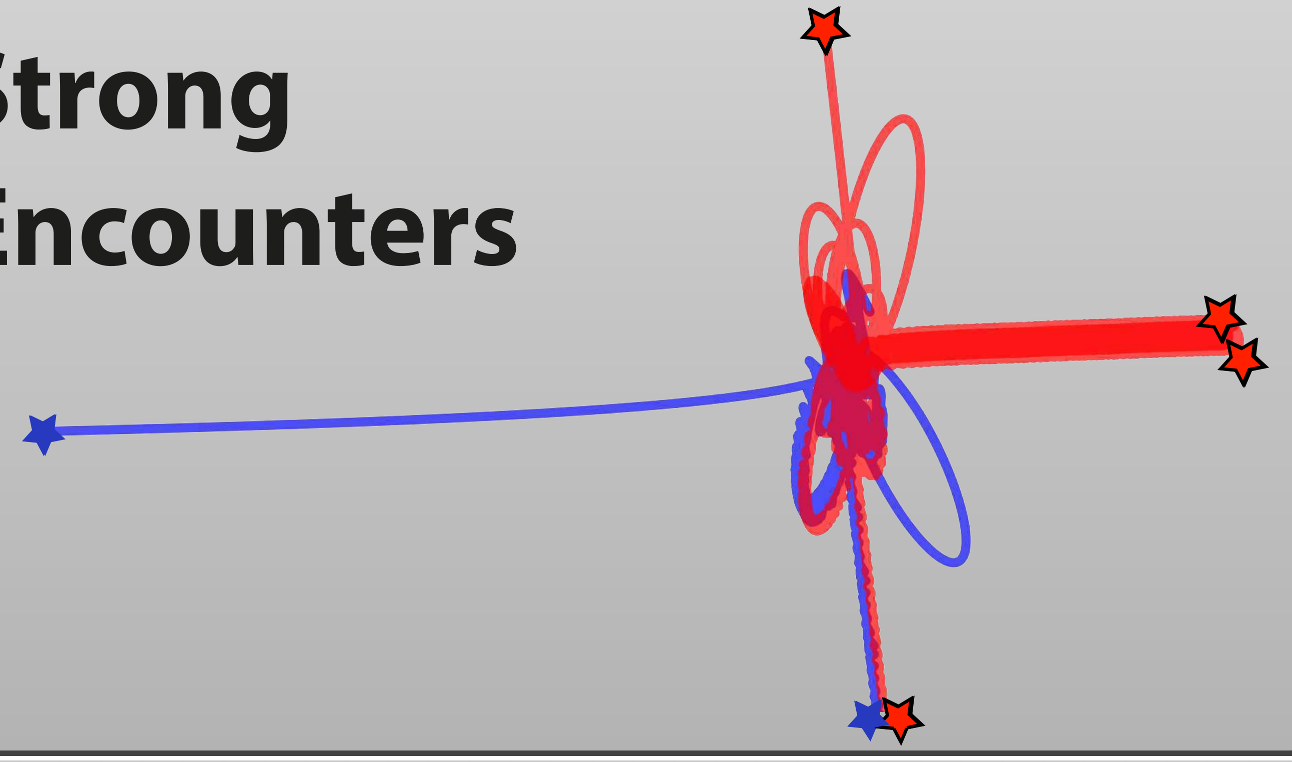
Hénon's method allows us to simulate massive, dense star clusters ($N \sim 10^7$) with **all the relevant physics**

Dynamical Interactions

Scatterings



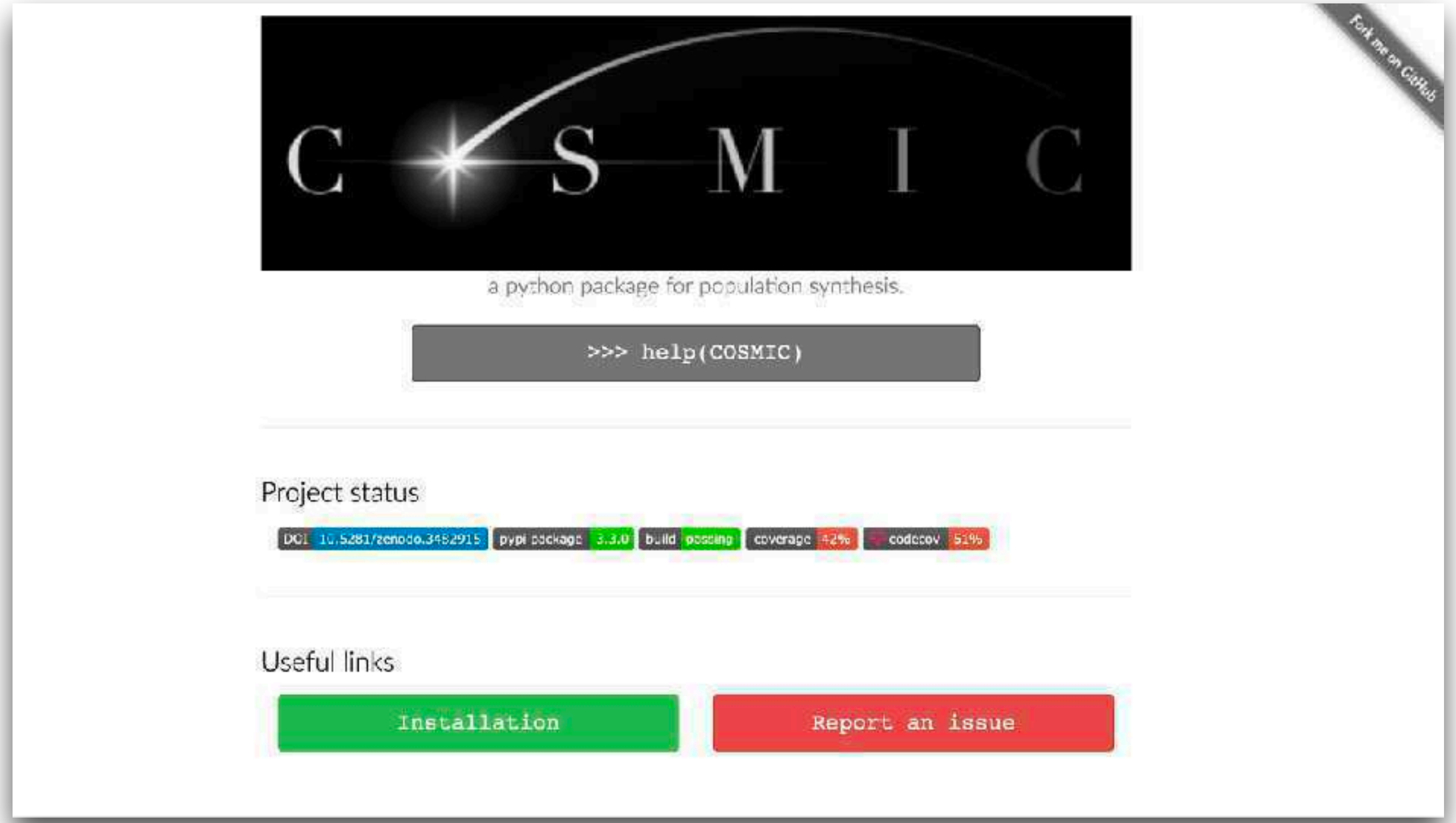
Strong Encounters



Stellar Evolution + Dynamics

Single and binary stellar evolution with **COSMIC**

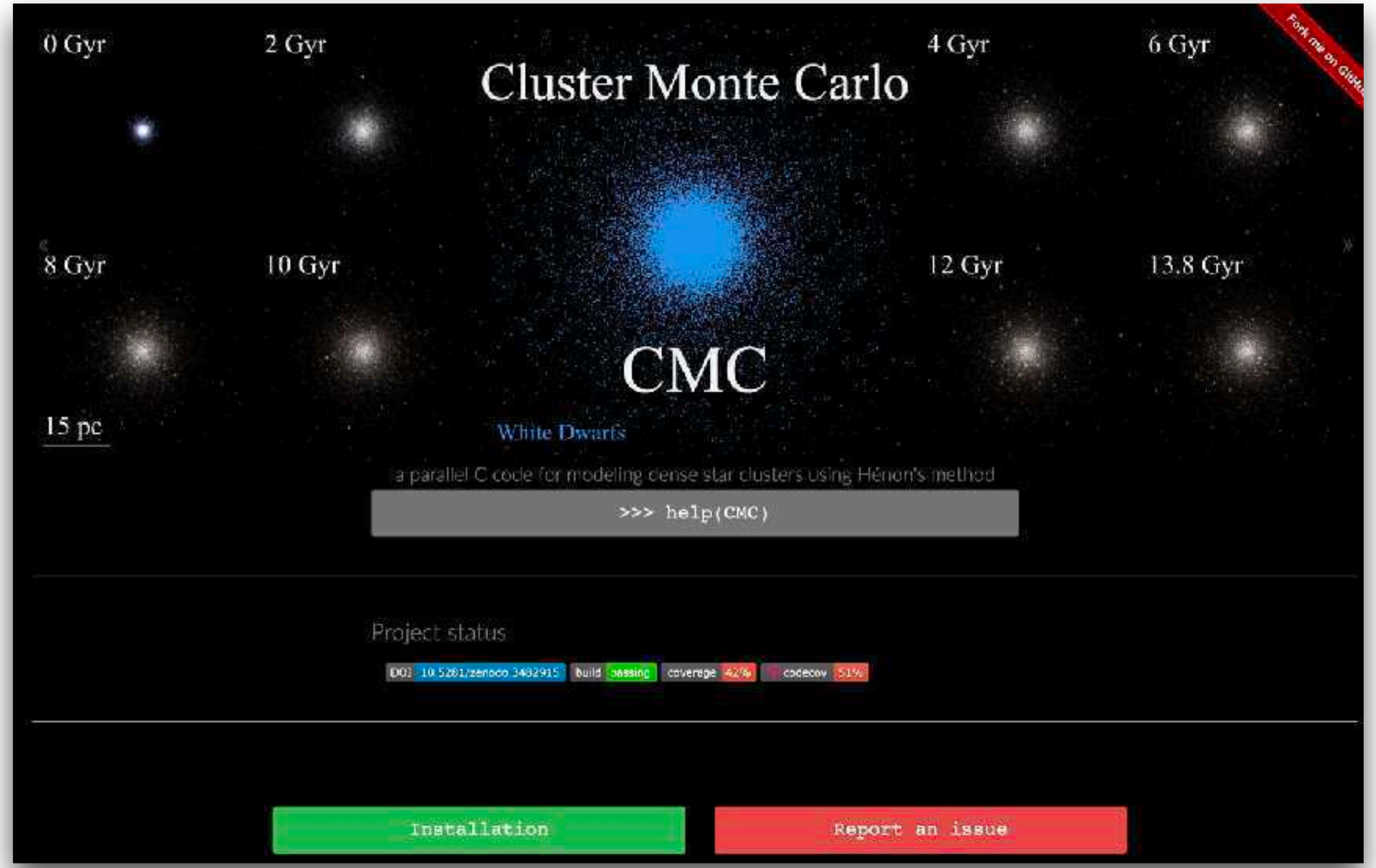
Breivik + CLR et al. (2020) ApJ, **898**, 71



https://cosmic-popsynth.github.io/

Dynamical Cluster Evolution with **CMC**

CLR et al. (2022) ApJSS, **258**, 2



https://clustermontecarlo.github.io/

Cool… does it work?

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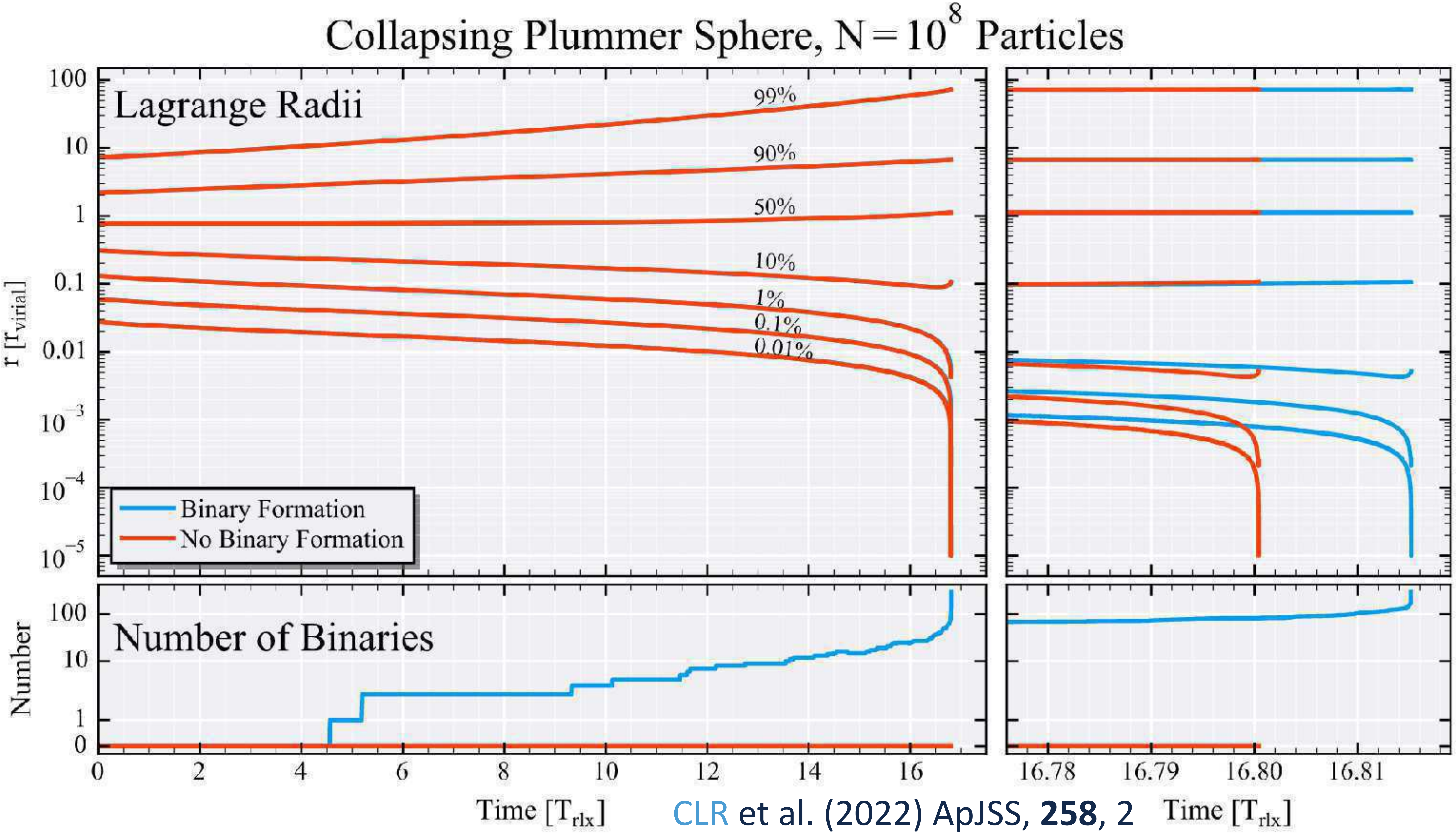

CLR et al. (2022) ApJSS, **258**, 2

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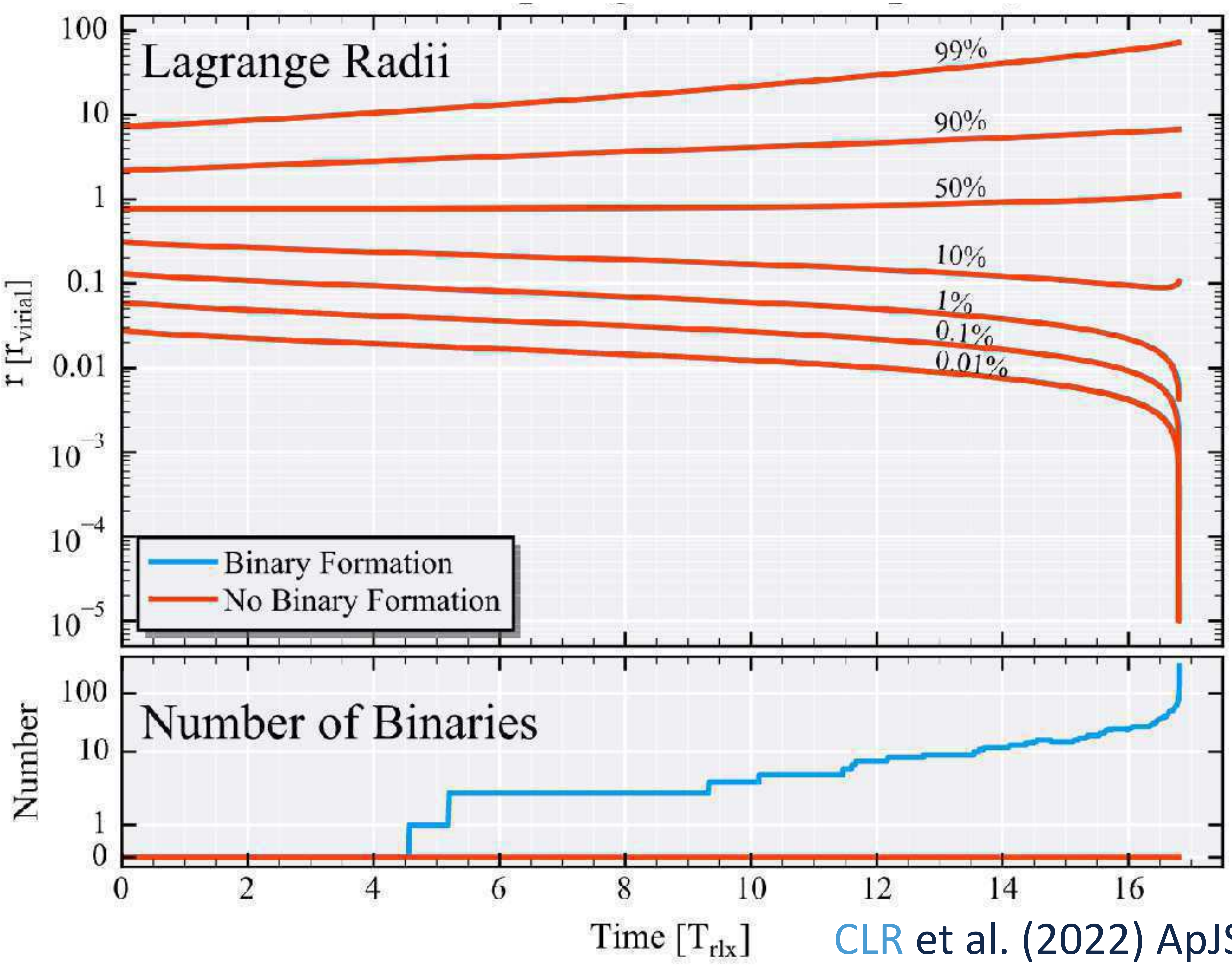

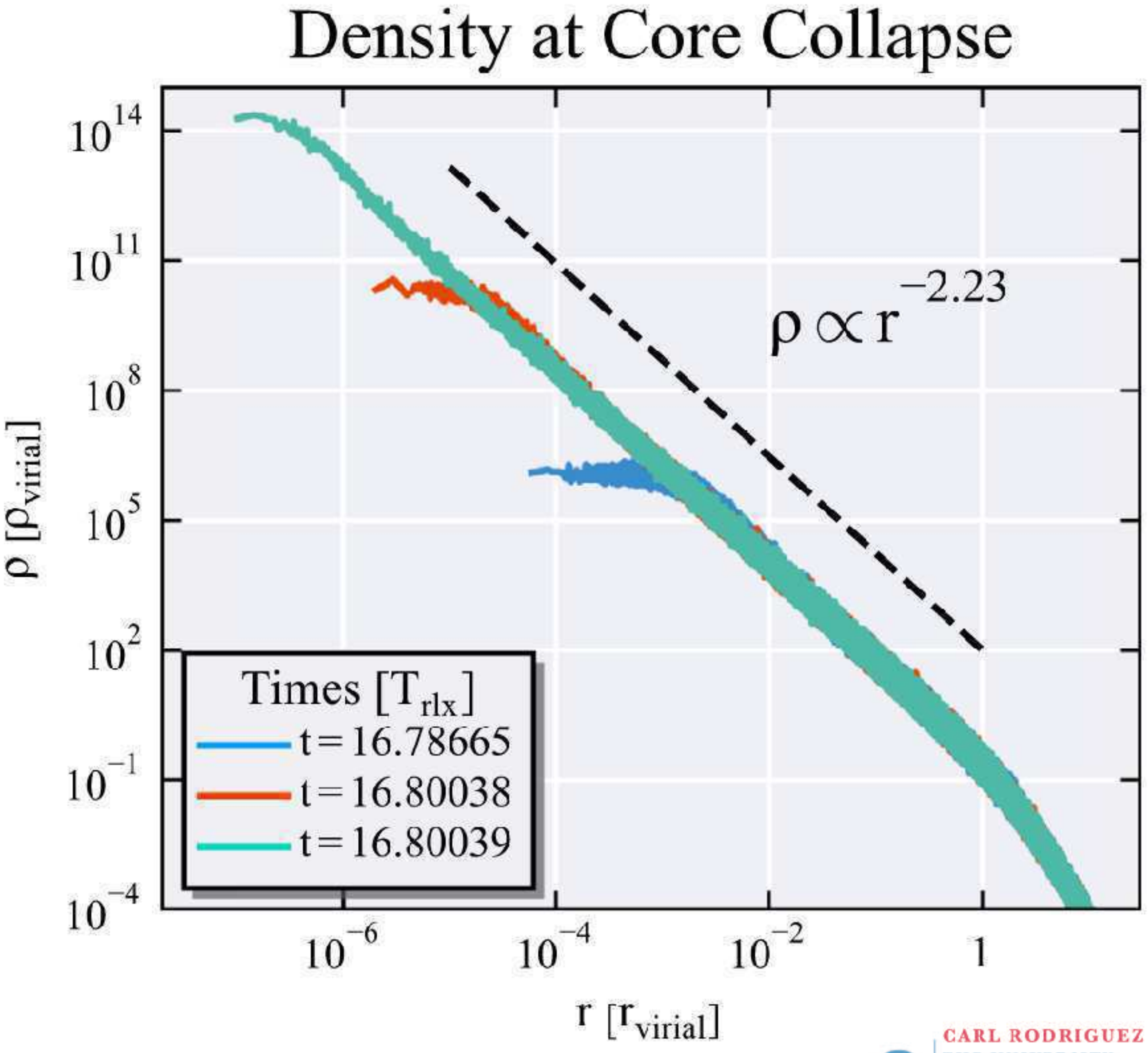

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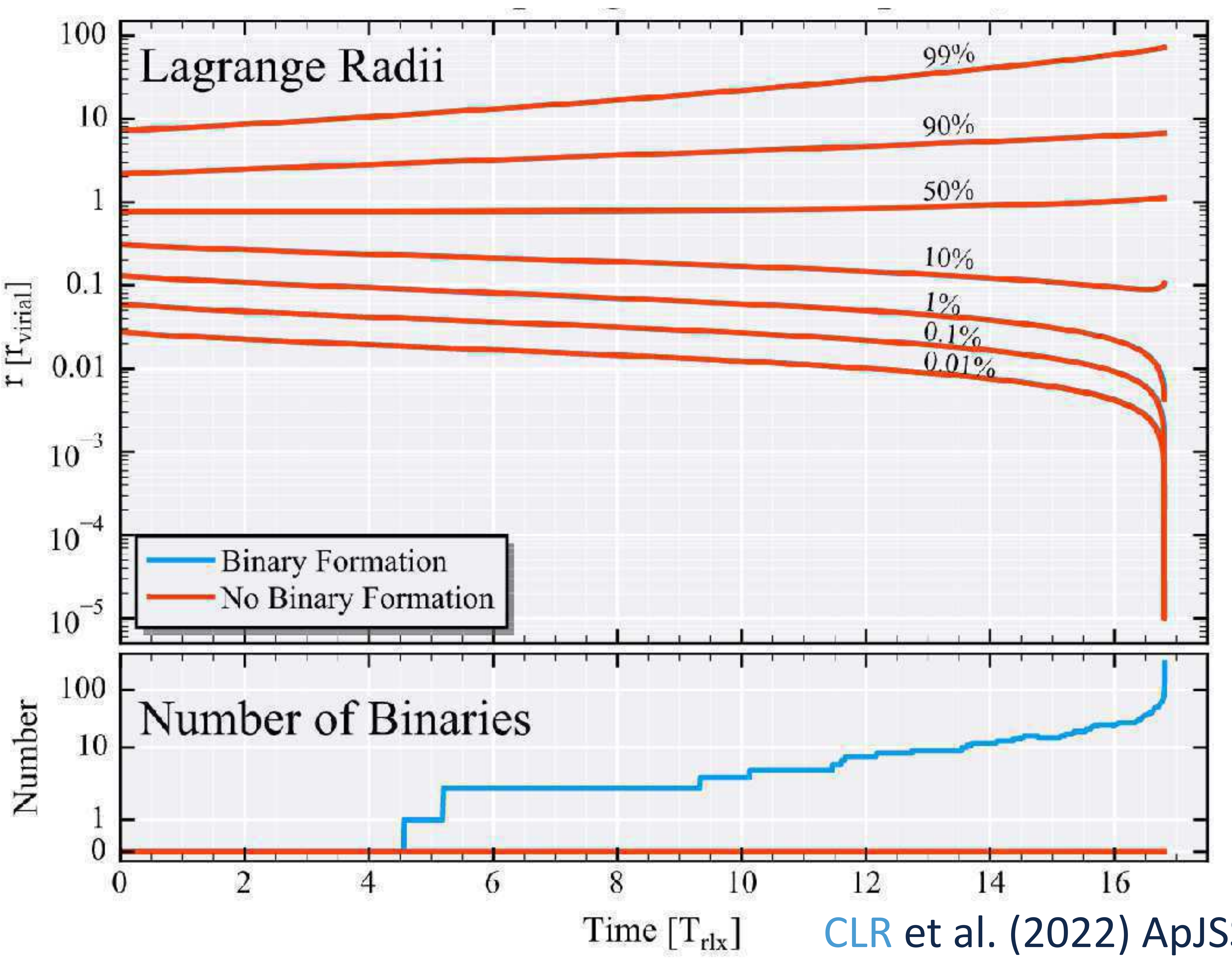

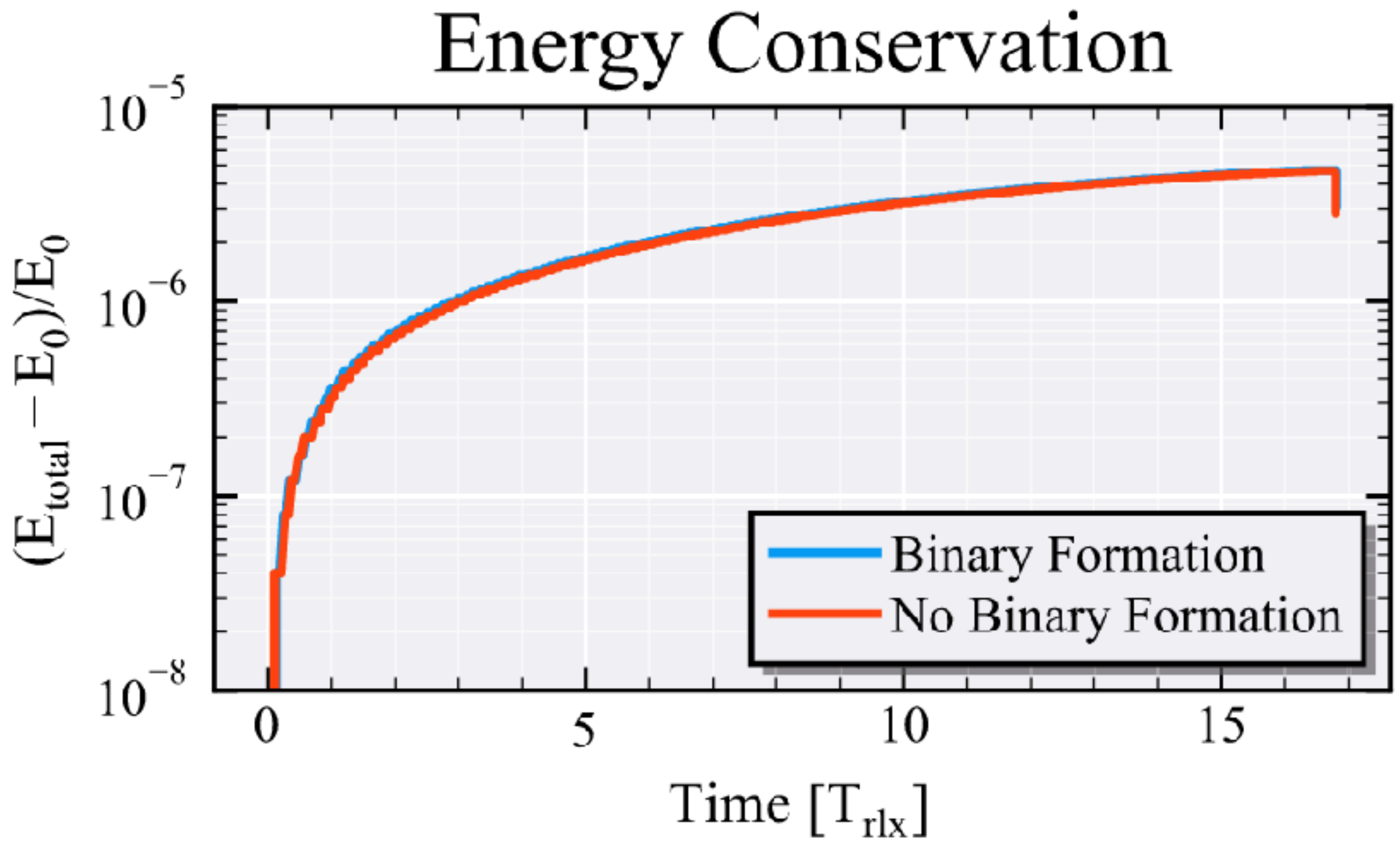

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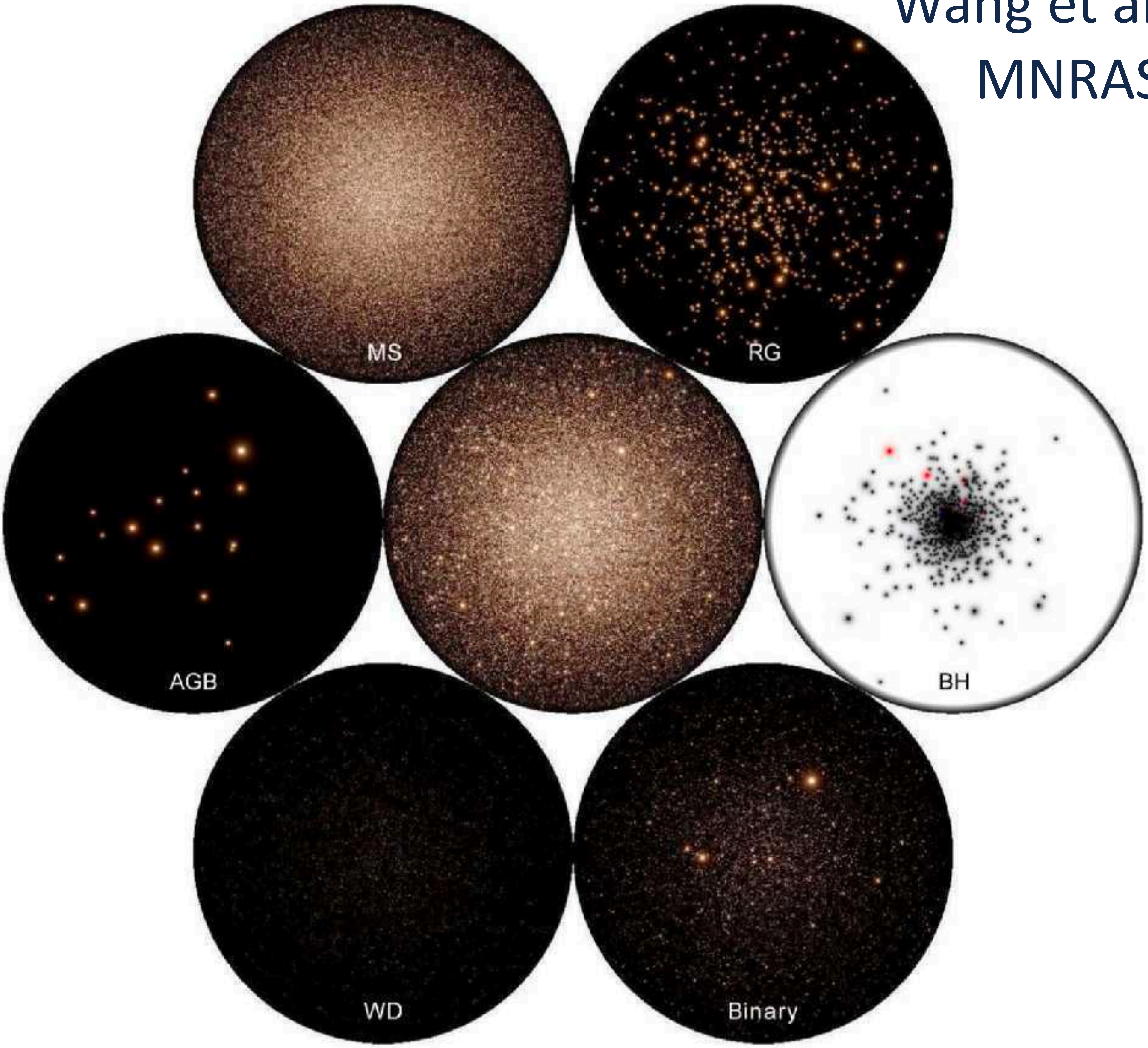
Wang et al. (2016)
MNRAS, **458**, 2



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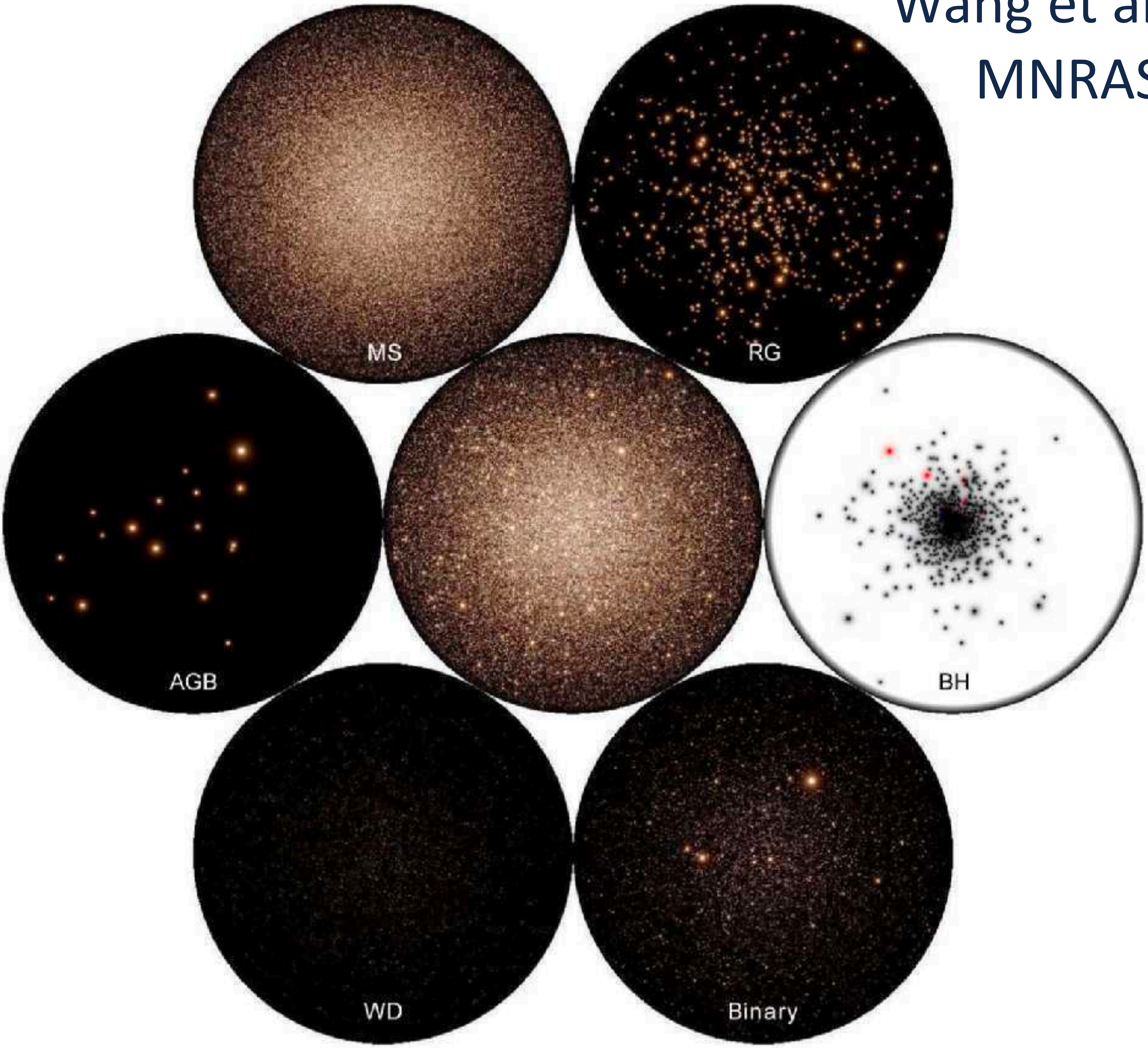

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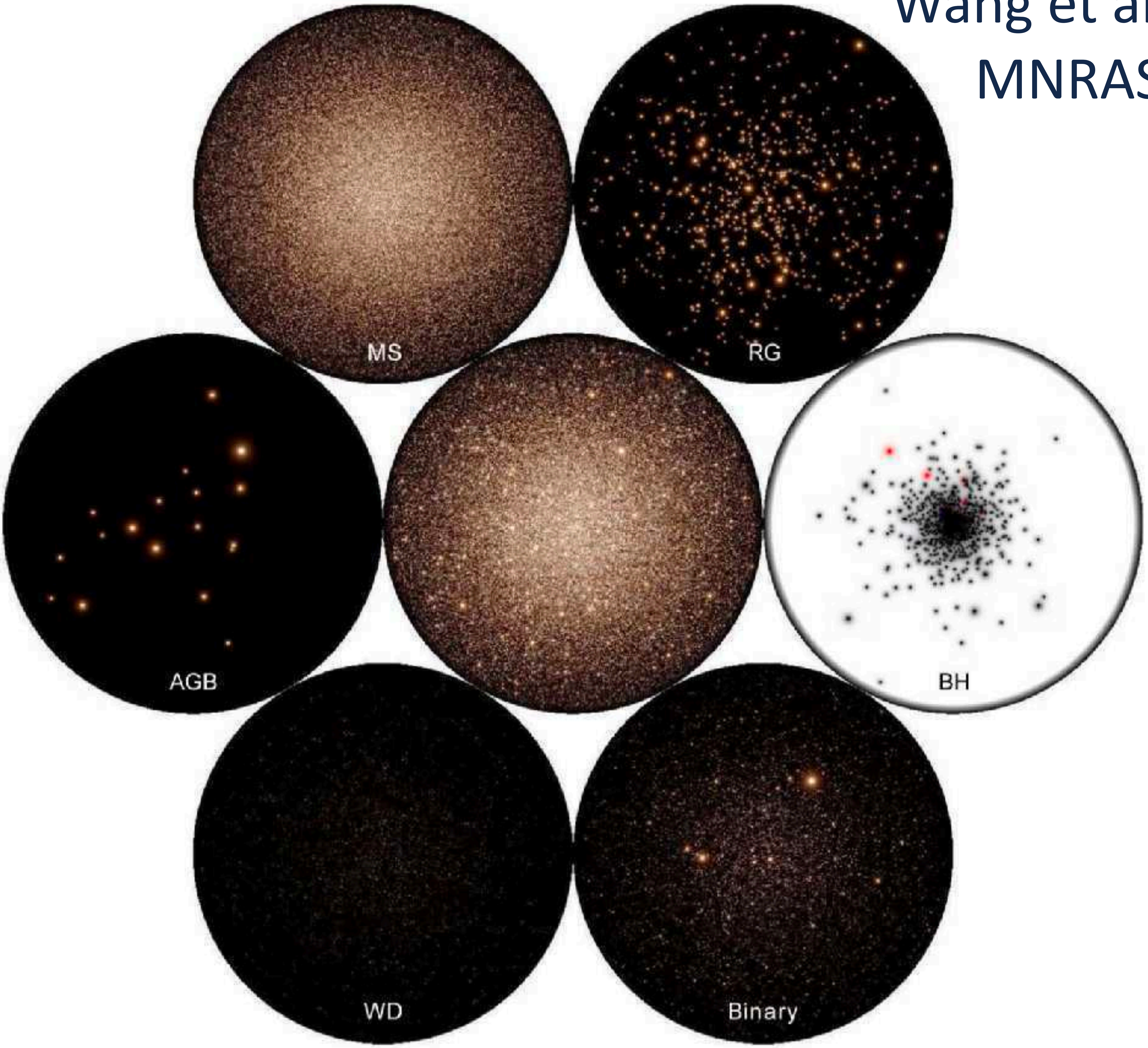

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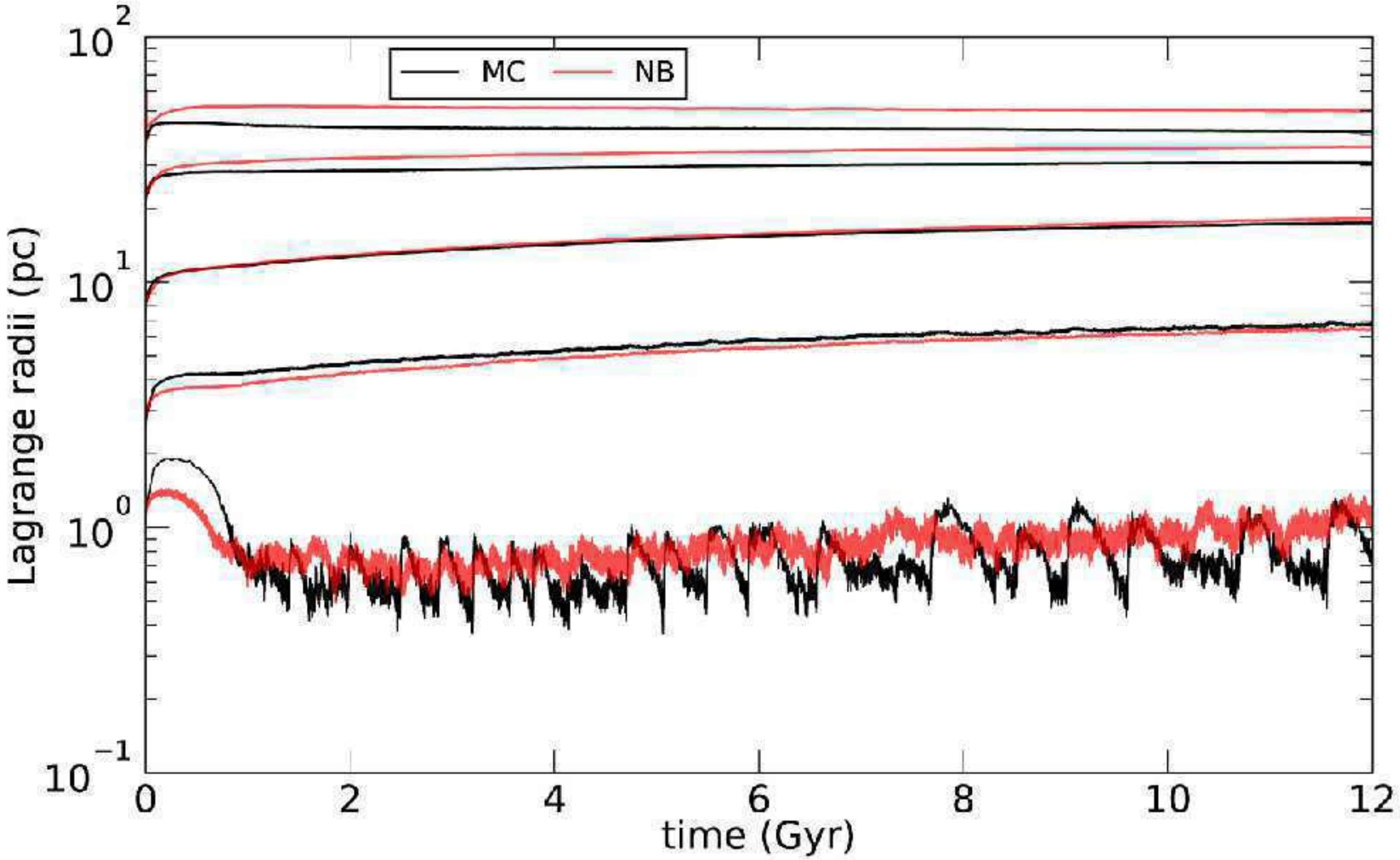
N-body ~ 13 **months** of wall time!
Monte Carlo ~ 2 days



CLR et al. (2022) MNRAS, **463**, 2109