"The original is unfaithful to the translation" ® ← J.L. Borges

Kinetic theory, from Plasmaish to Galacticese ® ← C. Hamilton + J.B. Fouvry, PoP review

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What is a galaxy? A collection of many stars w/ $\frac{1}{r}$ pot.

What is a plasma? A collection of many charges w/ $\frac{1}{r}$ pot.

To describe these,

$$\frac{d\vec{R}_i}{dt} = \frac{\partial H}{\partial \vec{V}_i} \quad \& \quad \frac{d\vec{V}_i}{dt} = -\frac{\partial H}{\partial \vec{R}_i} \quad \text{for each } i = 1 \dots N$$

$$\frac{\partial H}{\partial \vec{V}_i} \to \vec{V}_i \qquad -\frac{\partial H}{\partial \vec{R}_i} \to -\vec{\nabla}\Phi_g \text{ or } \frac{q_\alpha}{m_\alpha}\left(\vec{E} + \frac{\vec{V}_i}{c}\times\vec{B}\right)$$

Compute electromagnetic fields produced by the charged particles, or gravitational fields produced by the masses, then use these fields to evolve the particles' phase-space positions. Done! Well...

① $\sim 10^{28}$ molecules in this room. Double-precision data dump would be $\sim 10^{18}$ TB! ($\sim 10^{71}$ protons in ICM of a cluster of galaxies)

② Not really interested in every single particle.

③ Many-body systems are often chaotic — initial conditions are arbitrary, and infinitesimal changes to them yield macro~~sco~~scopically different results.

→ Statistical approach. What is prob. of a particle having $\vec{r}$ and $\vec{v}$ in some 6D interval $d^3\vec{r}\, d^3\vec{v}$? How does this probability evolve? Under what conditions is this probabilistic evolution accurate enough to yield reliable predictions for a single realization of the system?

→ KINETIC THEORY.

"We can measure the globule of matter and the distances bet. them, but Space plasm itself is incomputable." — Nabokov, Ada or Ardor.

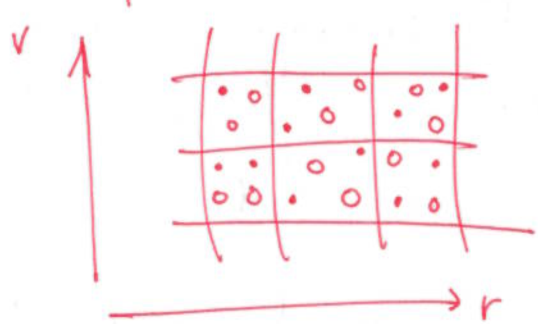
This works ~~when~~ because there is no difference statistically bet. flipping a coin $\sim 10^{28}$ times and flipping ~~—~~ $\sim 10^{28}$ coins one time each. Will say later why this is good for plasmas ...

Start w/ Klimontovich distribution:

$$F_\alpha(t, \vec{r}, \vec{v}) = \sum_{i=1}^{N_\alpha} \delta(\vec{r} - \vec{R}_{\alpha i}(t))\, \delta(\vec{v} - \vec{V}_{\alpha i}(t))$$

species label

$$\left(N_\alpha = \int d\vec{r} \int d\vec{v}\, F_\alpha \right)$$

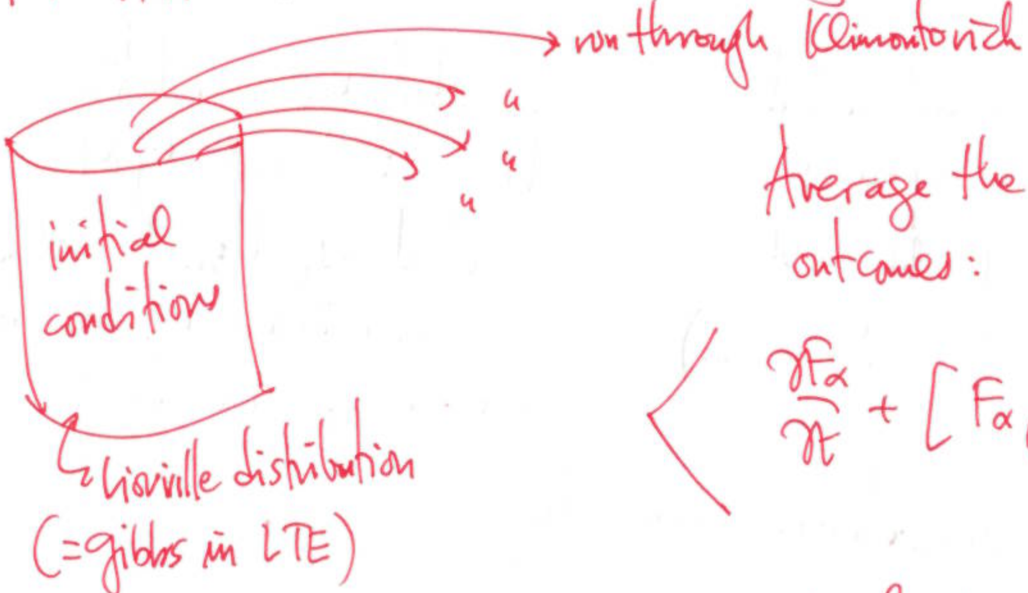


Can show that

$$0 = \frac{\partial F_\alpha}{\partial t} + \vec{v}\cdot\vec{\nabla} F_\alpha + \underbrace{\frac{q_\alpha}{m_\alpha}\left(\vec{E}_m + \frac{\vec{v}}{c}\times\vec{B}_m\right)\cdot\frac{\partial F_\alpha}{\partial \vec{v}}}_{[F_\alpha, H_m]}$$

(a Liouville thm. for point particles)

Trade 6N equs. to 1 eqn. with 6N dimensions.
It's still deterministic and nothing lost. So...



Average the deterministic outcomes:

$$\left\langle \frac{\partial F_\alpha}{\partial t} + [F_\alpha, H_m] = 0 \right\rangle_{P_N}$$

What is $\langle F_\alpha \rangle_{P_N}$? "one-part. dist. function": prob. that particle α_1 has coordinates in $\vec{X}_{\alpha_1}$ to $\vec{X}_{\alpha_1} + d\vec{X}_{\alpha_1}$ regardless of where all other particles are.

$$\Rightarrow f_\alpha = \langle F_\alpha \rangle = N_\alpha \int d\vec{X}_{\alpha_2}\, d\vec{X}_{\alpha_3} \ldots d\vec{X}_{\alpha N_\alpha}\, d\vec{X}_{\beta_1} \ldots d\vec{X}_{\beta N_\beta} \ldots P_N$$

Similarly, two-particle dist. func:

$$f_{\alpha\beta} = N_\alpha N_\beta \int d\vec{X}_{\alpha_2} \ldots d\vec{X}_{\alpha N_\alpha}\, d\vec{X}_{\beta 2} \ldots d\vec{X}_{\beta N_\beta} \ldots P_N$$

(explain)

$$\Rightarrow \frac{\partial f_\alpha}{\partial t} + \langle [F_\alpha, H_m] \rangle_{P_N} = 0$$

$$\langle [F_\alpha, H_m] \rangle_{P_N} \rightarrow [f_\alpha, \langle H_m \rangle] + \langle [\underbrace{\delta F_\alpha}_{F_\alpha - f_\alpha}, \underbrace{\delta H_m}_{H_m - \langle H_m \rangle}] \rangle$$

nonlinearity

$$\Rightarrow \underbrace{\frac{\partial f_\alpha}{\partial t} + [f_\alpha, \langle H_m \rangle]}_{\text{1-part. dist. advected through phase space by mean field (still must justify what "mean" means...)}} = - \underbrace{\langle [\delta F_\alpha, \delta H_m] \rangle}_{\text{affected by correlations bet. } \delta F_\alpha \text{ and E\&M fluctuations, both of which are associated w/ discrete nature of charges.}}$$

Take electrostatic interactions... can show

$$\frac{\partial f_\alpha}{\partial t} + \vec{v}\cdot\vec{\nabla} f_\alpha + \frac{q_\alpha}{m_\alpha}\left[\vec{E} + \frac{\vec{v}}{c}\times\vec{B}\right]\cdot\frac{\partial f_\alpha}{\partial \vec{v}}$$

$$= -\Sigma_\beta \frac{q_\alpha}{m_\alpha}\int d\vec{x}' \underbrace{\frac{\partial}{\partial \vec{r}}\frac{q_\beta}{|\vec{r}-\vec{r}'|}}_{-\vec{\nabla}\varphi} \cdot \frac{\partial}{\partial \vec{v}}\Big[f_\alpha(t,\vec{x}) f_\beta(t,\vec{x}') - f_{\alpha\beta}(t,\vec{x},\vec{x}')\Big]$$

If all particles are completely uncorrelated, then $f_\alpha f_\beta = f_{\alpha\beta}$

$\Rightarrow$ Vlasov (write it)

Write $f_{\alpha\beta} = f_\alpha f_\beta + g_{\alpha\beta}$ ← 2-part. correlation function (explain)

$$\Rightarrow \boxed{\dot{f}_\alpha = \Sigma_\beta \frac{q_\alpha}{m_\alpha}\int d\vec{x}' \frac{\partial}{\partial \vec{r}}\frac{q_\beta}{|\vec{r}-\vec{r}'|}\cdot\frac{\partial}{\partial \vec{v}} g_{\alpha\beta}}$$

1st step of BBGKY hierarchy (Bogoliubov, Born, Green, Kirkwood, Yvon: 1935–1949)

Evolution of 1-part. DF depends on correlations bet. 2 part's. Relationships bet. 2 discrete particles affect future of each of them.

Can write eqn. for $g_{\alpha\beta}$:

$$\left(\frac{\partial}{\partial t} + v\cdot\nabla + a\cdot\frac{\partial}{\partial v} + v'\cdot\nabla' + a'\cdot\frac{\partial}{\partial v'}\right) g_{\alpha\beta}$$

(cons. move through phase space, pushed by smooth fields)

$$= \Sigma_\gamma \frac{q_\alpha}{m_\alpha}\int d\vec{x}'' \frac{\partial}{\partial r}\frac{q_\gamma}{|r-r''|}\cdot\frac{\partial f_\alpha}{\partial v}\, g_{\beta\gamma}$$

(α forms relationships w/ the bath — will be Debye shielding) — affects $\alpha\beta$ relationship

$$+ \Sigma_\gamma \frac{q_\beta}{m_\beta}\int dx'' \frac{\partial}{\partial r'}\frac{q_\gamma}{|r'-r''|}\cdot\frac{\partial f_\beta}{\partial v'}\, g_{\alpha\gamma}$$

(β forms relationships w/ the bath — affects $\alpha\beta$ relationship)

$$+ \frac{\partial}{\partial r}\frac{q_\alpha q_\beta}{|r-r'|}\cdot\left(\frac{1}{m_\alpha}\frac{\partial}{\partial v} - \frac{1}{m_\beta}\frac{\partial}{\partial v'}\right)\left(f_\alpha f_\beta + g_{\alpha\beta}\right)$$

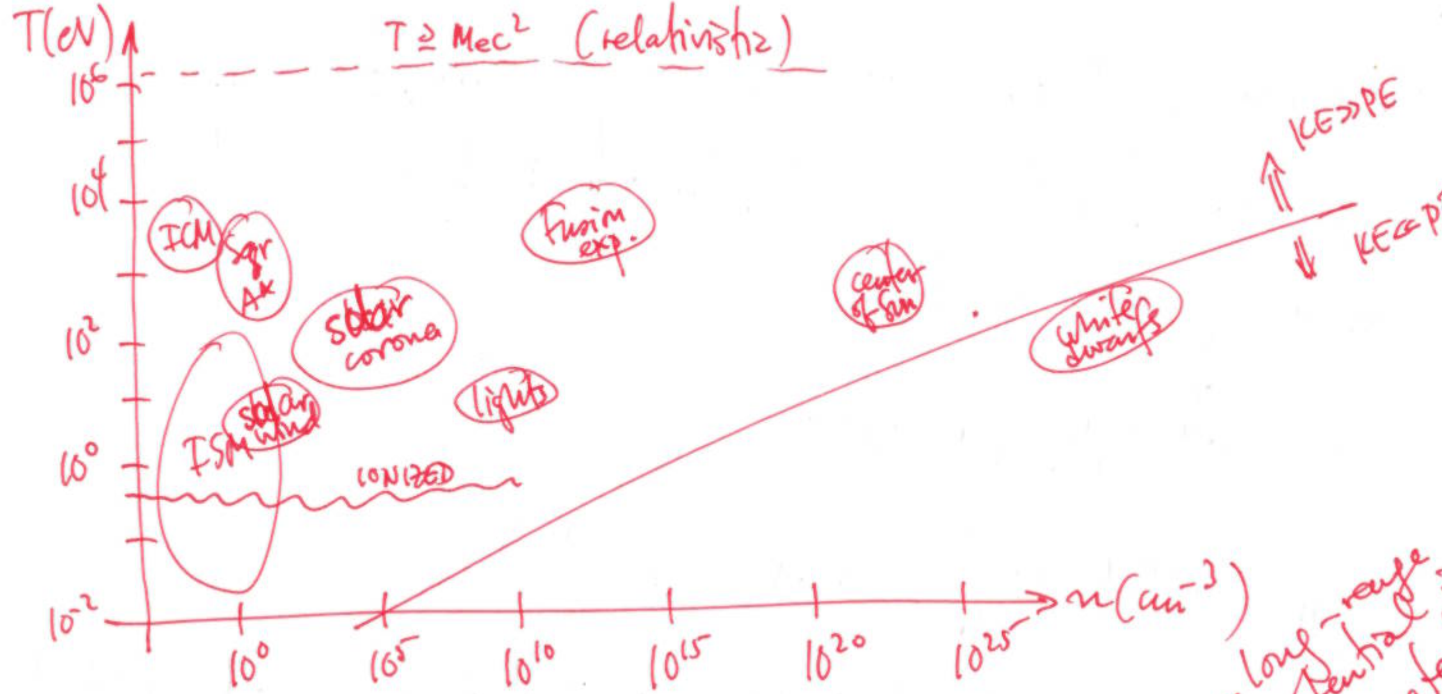
existing correlations continue to source themselves.

(2 initially uncorrelated part's ~~from~~ form a relationship)

$$+ \Sigma_\gamma q_\gamma \int dx''\left(\frac{1}{m_\alpha}\frac{\partial}{\partial r}\frac{q_\alpha}{|r-r''|}\cdot\frac{\partial}{\partial v} + \frac{1}{m_\beta}\frac{\partial}{\partial r'}\frac{q_\beta}{|r'-r''|}\cdot\frac{\partial}{\partial v'}\right) h_{\alpha\beta\gamma}$$

(ménage à trois)

Keeps going... fortunately, straightforward to close for a plasma...



long-range potential is attenuated

When KE>>PE, BBGKY in equilibrium gives

$$f_{\alpha\beta} = \underbrace{f_\alpha(\vec{v}) f_\beta(\vec{v})}_{\text{Maxwellians}} \left[1 - \underbrace{\frac{q_\alpha q_\beta}{T} \frac{e^{-r/\lambda_D}}{r}}_{\sim \frac{1}{\Lambda} \cdot \frac{e^{-x}}{x}} \right] \quad \text{where } \lambda_D^2 \equiv \left(\Sigma_\alpha \frac{4\pi q_\alpha^2 n_\alpha}{T} \right)^{-1}$$

w/ $\Lambda = n \lambda_D^3 \ggg 1$.

$\Rightarrow$ "quasi-neutral" on scales $\gg \lambda_D \cong 7\,m \left(T_{eV}/n_{cm^{-3}} \right)^{1/2}$. $\Lambda \sim \frac{d_{interp}}{\lambda_{Debye}}$

* tells us that correlations are weak for $l \gg \lambda_D$, and particles are almost distributed uniformly.

leads to "Bogoliubov's timescale hierarchy":

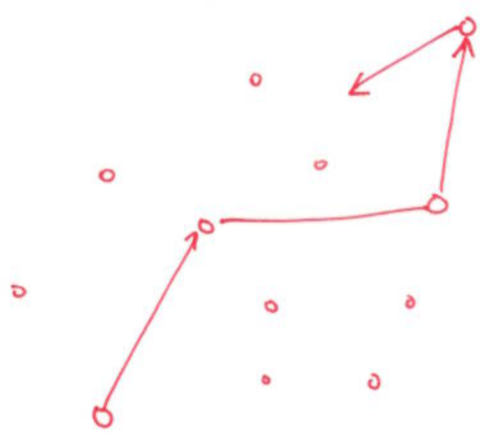
- On lengthscales $\sim \lambda_D$ and timescales $\sim v_{th}^{-1} \lambda_D$, pair correlations are established and Coulomb potentials are shielded. (This is a reversible process.)

- On lengthscales $\sim \lambda_{mfp} \sim v_{th}/\nu \sim \lambda_D \left(\frac{\Lambda}{\ln\Lambda}\right)$ and timescales $\sim \nu^{-1} \sim \omega_p^{-1}\left(\frac{\Lambda}{\ln\Lambda}\right)$, local Maxwellian VDFs are attained via collisional relaxation.

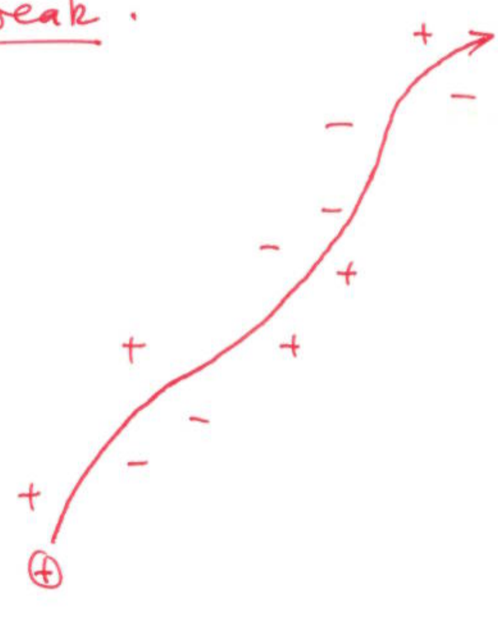
- Macroscopic dynamics on lengthscales $\sim L$ and timescales $\sim u^{-1}L \sim \left(\frac{v_{th}}{u}\right)\nu^{-1}\left(\frac{L}{\lambda_{mfp}}\right) \gg \nu^{-1}$ or $\gtrsim \nu^{-1}$

  "collisional" (fluid) — "weakly collisional" or "collisionless"

This is completely different from the situation of a gas of neutral particles. There, the interaction range is $\sim r_0 \ll$ mean spacing $\sim n^{-1/3} \Rightarrow n r_0^3 \lll 1 \Rightarrow$ collisions are *rare*. By contrast, in a plasma, $r_0 \approx \lambda_D \ggg n^{-1/3} \Rightarrow n r_0^3 \ggg 1 \Rightarrow$ collisions (actually, correlations) happen all the time... NOT *rare*. Even though a charged particle is interacting w/ all particles in its Debye sphere and thus undergoes $\sim \Lambda$ simultaneous interactions, *such interactions are weak*.



vs.



straight-line trajectories

① Assume f_α is homogeneous on scale over which $g_{\alpha\beta}$ varies. (Good b/c $\Lambda >>> 1$)

② Assume particles enter into relationships initially unacquainted (Markov assumption)

③ Drop 3-body correlations (weak — not rare)

$$\Rightarrow \left(\frac{\partial f_\alpha}{\partial t}\right)_{\text{Vlasov}} = \sum_\beta \int \frac{d\vec{k}}{(2\pi)^3}\, \frac{\vec{k}}{m_\alpha}\cdot\frac{\partial}{\partial \vec{v}} \int d\vec{v}' \left|\frac{4\pi q_\alpha q_\beta}{k^2 D(\vec{k}\cdot\vec{v}, \vec{k})}\right|^2$$

$$\times\ \pi\, \delta(\vec{k}\cdot\vec{v} - \vec{k}\cdot\vec{v}') \left(\frac{\vec{k}}{m_\alpha}\cdot\frac{\partial}{\partial \vec{v}} - \frac{\vec{k}}{m_\beta}\cdot\frac{\partial}{\partial \vec{v}'}\right) f_\alpha(\vec{v}) f_\beta(\vec{v}')$$

$$\equiv \left(\frac{\partial f_\alpha}{\partial t}\right)_{c,\,B\text{-}L}$$

2 initially uncorrelated particles...

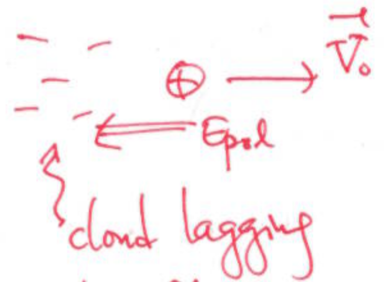
... push equal and oppositely on one another in direction of their Coulomb electric fields...

$\omega - k\cdot v = 0$

... interaction requires resonance (Landau-resonant excitation of waves, Landau-resonant absorption of waves)

dressed potential: Coulomb $\propto \frac{1}{k^2}$ shielding by equilibrated Debye cloud in frame of particle

Explain polarization drag via pair correlation est'd on $\sim \omega_p^{-1}$ time



~~Decel.~~ Deceleration sources waves... these contribute to "thermal bath" off of which other particles can diffuse

New Debye cloud after every deflection of charge

Because $\nu \ll \omega_p$, charges always carry well-equilibrated Debye clouds w/ them.

(Rostoker test-particle superposition principle: plasma composed of uncorrelated Debye clouds. $\Rightarrow$ Balescu-Lenard)

In equilibrium, minimum level of Electric noise given by

$$W = \int \frac{d\vec{k}}{(2\pi)^3} \underbrace{\frac{T}{2}}_{T/2 \text{ per degree of freedom}} \frac{1}{1+(k/k_D)^2}$$

; fluctuation-dissipation relation

$$\Big(\underbrace{P_{\omega,k}}_{\text{power lost by polarization drag}} = \underbrace{-2\omega W_{\omega,k} \operatorname{Im}\left[D(\omega,k)\right]}_{\text{power in emitted } \cancel{\text{and}} \text{ Landau-damped waves, which } \cancel{\text{are}} \text{ is absorbed by other resonant particles}} \Big)$$

BL in FP form: $\frac{\partial f_\alpha}{\partial t} = -\frac{\partial}{\partial \vec{v}} \cdot \left[\vec{A} f_\alpha \right] + \frac{1}{2} \frac{\partial}{\partial \vec{v}} \frac{\partial}{\partial \vec{v}} : \left[\overleftrightarrow{B} f_\alpha \right]$

$\vec{A}$: $\frac{q_\alpha}{m_\alpha} \vec{E}_{pol} + \frac{1}{2} \frac{\partial}{\partial \vec{v}} \cdot \overleftrightarrow{B}_\alpha(\vec{v})$

(from imperfect Debye shielding)

$\overleftrightarrow{B}$: $\left(\frac{q_\alpha}{m_\alpha} \right)^2 \left\langle \delta\vec{E}\, \delta\vec{E}^* \right\rangle_{\omega = k \cdot v}$

(from resonant diffusion off quasi-thermal bath)

This FP form confirms that scatterings are small angle.

OR

$-\frac{\partial}{\partial \vec{v}} \cdot \left[\frac{q_\alpha}{m_\alpha} \vec{E}_{pol} \right] + \frac{1}{2} \frac{\partial}{\partial \vec{v}} \cdot \left[\overleftrightarrow{B} \cdot \frac{\partial f_\alpha}{\partial \vec{v}} \right]$

Too complicated to implement in plasma calculations — no one ever does this. But it does spell out the path forward for understanding other stochastic processes, e.g., diffusion/drag in a spectrum of (longer-scale) waves ("quasi-linear operator"), and it is useful for computing spatial diffusion coefficients related to collisional relaxation of temperature and flow gradients (Chapman - Enskog - Braginskii).