lintheorychap

6
Fluctuations II: linear response theory

In the last chapter we made extensive use of the linearised CBE:

linCBEaa

$$\frac{\partial f_1}{\partial t} + \boldsymbol{\Omega}\cdot\frac{\partial f_1}{\partial \boldsymbol{\theta}} = \frac{\partial f_0}{\partial \mathbf{J}}\cdot\frac{\partial \Phi_1}{\partial \boldsymbol{\theta}} \tag{6.1}$$

and we recognised the need to make explicit the linear dependence of the perturbed potential $\Phi_1(\mathbf{x})$ on the perturbed DF $f_1(\boldsymbol{\theta},\mathbf{J})$. This chapter starts by achieving this goal and goes on to use this achievement to discuss the small-amplitude oscillations of stellar systems. We begin by introducing a crucial piece of technology.

We will be making extensive use of Poisson's equation, so it is very important to be clear that in this chapter f is normalised such that the mass density $\rho = \int \mathrm{d}^3\mathbf{v}\, f$.

Biorthosec

6.1 Biorthogonal potential-density pairs

Biorthogonal potential-density pairs
biorthogonal potential-density!

A basis of **biorthogonal potential-density** pairs is a set of pairs of functions on real space, $(\rho^{(\alpha)}, \Phi^{(\alpha)})$, such that

Poisson

$$4\pi G\rho^{(\alpha)} = \nabla^2\Phi^{(\alpha)} \quad \text{and} \quad \int \mathrm{d}^3\mathbf{x}\,\Phi^{(\alpha)*}\rho^{(\alpha')} = -\mathcal{E}\delta_{\alpha\alpha'}, \tag{6.2}$$

where $\mathcal{E}$ is a constant with the dimensions of energy. The subscript α usually stands for a triple of integers: it is often convenient for $\Phi^{(\alpha)}$ to be a product of a function $g(r)$ times a spherical harmonic $Y_\ell^m(\vartheta,\phi)$ and in this case $\alpha=(n,\ell,m)$, where $g(r)$ depends on the 'radial quantum number' n and the 'total angular-momentum quantum number' ℓ. Appendix D explains how sets of potential-density pairs are constructed and gives some specific examples.

Given a density distribution $\rho(\mathbf{x})$, we expand it in our basis as

defsAp
$$\rho(\mathbf{x})=\sum_\alpha A_\alpha\rho^{(\alpha)}(\mathbf{x}) \quad\Rightarrow\quad \begin{cases}\Phi(\mathbf{x})=\sum_\alpha A_\alpha\Phi^{(\alpha)}(\mathbf{x}),\\ A_\alpha=-\dfrac{1}{\mathcal{E}}\displaystyle\int \mathrm{d}^3\mathbf{x}\,\Phi^{(\alpha)*}(\mathbf{x})\rho(\mathbf{x}).\end{cases} \tag{6.3}$$

If ρ and Φ are time-dependent, the A_α become time-dependent. Combining the equations on the right of (6.3) we have

PhiA
$$\Phi(\mathbf{x})=-\frac{1}{\mathcal{E}}\int \mathrm{d}^3\mathbf{x}'\sum_\alpha \Phi^{(\alpha)*}(\mathbf{x}')\Phi^{(\alpha)}(\mathbf{x})\rho(\mathbf{x}'). \tag{6.4}$$

Thus biorthogonal pairs provide a means of solving Poisson's equation for the potential generated by any density distribution. Comparing equation (6.4) with Poisson's integral, we see that

PoissonAA
$$\frac{G}{|\mathbf{x}'-\mathbf{x}|}=\frac{1}{\mathcal{E}}\sum_\alpha \Phi^{(\alpha)*}(\mathbf{x}')\Phi^{(\alpha)}(\mathbf{x}). \tag{6.5}$$

When the disturbed density ρ_1 is specified by the disturbed DF f_1, the coefficients of the disturbed density are given by

Afromfk
$$\begin{aligned}A_\alpha&=-\frac{1}{\mathcal{E}}\int \mathrm{d}^6\mathbf{w}\,\Phi^{(\alpha)*}f_1=-\frac{1}{\mathcal{E}}\int \mathrm{d}^3\mathbf{J}\mathrm{d}^3\boldsymbol{\theta}\sum_{\mathbf{k}}\Phi^{(\alpha)*}_{\mathbf{k}}\mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot\boldsymbol{\theta}}\sum_{\mathbf{k}'}f_{\mathbf{k}}\mathrm{e}^{\mathrm{i}\mathbf{k}'\cdot\boldsymbol{\theta}}\\ &=-(2\pi)^3\frac{1}{\mathcal{E}}\int \mathrm{d}^3\mathbf{J}\sum_{\mathbf{k}}\Phi^{(\alpha)*}_{\mathbf{k}}f_{\mathbf{k}}.\end{aligned} \tag{6.6}$$

We'll need this equation below.

The expansion coefficients A_α are important descriptors of the system. For example, its potential energy is

$$P=\tfrac{1}{2}\int \mathrm{d}^3\mathbf{x}\,\Phi\rho=\tfrac{1}{2}\int \mathrm{d}^3\mathbf{x}\,\Phi^*\rho=\tfrac{1}{2}\int \mathrm{d}\mathbf{x}\sum_\alpha A^*_\alpha\Phi^{(\alpha)*}\rho=-\tfrac{1}{2}\mathcal{E}\sum_\alpha|A_\alpha|^2. \tag{6.7}$$

A related quantity is the potential energy that a system with density $\rho(\mathbf{x})$ has by virtue of sitting in the potential generated by another system $\widetilde{\rho}(\mathbf{x})$. This is

$$\begin{aligned}\widetilde{P}&=-G\int \mathrm{d}^3\mathbf{x}\,\mathrm{d}^3\mathbf{x}'\frac{\rho(\mathbf{x})\widetilde{\rho}(\mathbf{x}')}{|\mathbf{x}'-\mathbf{x}|}\\ &=-\frac{1}{\mathcal{E}}\sum_\alpha\int \mathrm{d}^3\mathbf{x}\,\mathrm{d}^3\mathbf{x}'\rho(\mathbf{x})\Phi^{(\alpha)*}(\mathbf{x})\Phi^{(\alpha)}(\mathbf{x}')\widetilde{\rho}(\mathbf{x}')=-\mathcal{E}\sum_\alpha A_\alpha\widetilde{A}^*_\alpha.\end{aligned} \tag{6.8}$$

EofFsec

6.2 Energy of a disturbance

Energy of a disturbance

All disturbances of a stable system have positive energy, so work has to be done to disturb the system. If there's a disturbance with zero energy, growth of its amplitude is consistent with conservation of energy and the system is liable to be unstable.

We compute the energy associated with a disturbance f_1 to the DF by calculating the work we need to do via an external potential Φ_e to excite it. The external field applies a force $\mathbf{F} = -\nabla\Phi_\mathrm{e}(\rho_1 \mathrm{d}^3\mathbf{x})$ to the excess mass $\rho_1 \mathrm{d}^3\mathbf{x}$, which is moving at velocity $\langle\mathbf{v}\rangle$, so the rate at which the external field works is

rawdEdt
$$\frac{\mathrm{d}E}{\mathrm{d}t} = -\int \mathrm{d}^3\mathbf{x}\ \langle\mathbf{v}\rangle\cdot\nabla\Phi_\mathrm{e}\int \mathrm{d}^3\mathbf{v}\, f_1 = -\int \mathrm{d}^6\mathbf{w}\, f_1\mathbf{v}\cdot\nabla\Phi_\mathrm{e}. \tag{6.9}$$

Now

$$\mathbf{v}\cdot\nabla\Phi_\mathrm{e} = -[H_0,\Phi_\mathrm{e}] = \boldsymbol{\Omega}\cdot\frac{\partial\Phi_\mathrm{e}}{\partial\boldsymbol{\theta}}, \tag{6.10}$$

where we've used $H_0(\mathbf{J})$, so equation (6.9) can be written

dEdtFourier
$$\frac{\mathrm{d}E}{\mathrm{d}t} = -\int \mathrm{d}^6\mathbf{w}\, f_1\boldsymbol{\Omega}\cdot\frac{\partial\Phi_\mathrm{e}}{\partial\boldsymbol{\theta}} = -(2\pi)^3\mathrm{i}\int \mathrm{d}^3\mathbf{J}\sum_{\mathbf{k}} f^*_{\mathbf{k}}\,\boldsymbol{\Omega}\cdot\mathbf{k}\,\Phi_{\mathrm{e}\mathbf{k}}, \tag{6.11}$$

where the second equality is obtained by expressing both f_1 and Φ_e as Fourier series in the angle variables and executing the integral over $\boldsymbol{\theta}$.

Substituting $\Phi_1 = \Phi_\mathrm{e} + \Phi_\mathrm{s}$ into equation (6.1), where Φ_s is the potential generated by f_1, we obtain

CBEsplit
$$\frac{\partial f_{\mathbf{k}}}{\partial t} + \mathrm{i}\mathbf{k}\cdot\boldsymbol{\Omega} f_{\mathbf{k}} - \mathrm{i}\mathbf{k}\cdot\frac{\partial f_0}{\partial\mathbf{J}}\Phi_{\mathrm{s}\mathbf{k}} = \mathrm{i}\mathbf{k}\cdot\frac{\partial f_0}{\partial\mathbf{J}}\Phi_{\mathrm{e}\mathbf{k}}. \tag{6.12}$$

When we use equation (6.12) to eliminate $\Phi_{\mathrm{e}\mathbf{k}}$ from equation (6.11), we obtain

threeterms
$$\frac{\mathrm{d}E}{\mathrm{d}t} = -(2\pi)^3\int \mathrm{d}^3\mathbf{J}\sum_{\mathbf{k}} f^*_{\mathbf{k}}\frac{\mathbf{k}\cdot\boldsymbol{\Omega}}{\mathbf{k}\cdot\nabla_{\mathbf{J}} f_0}\left(\frac{\partial f_{\mathbf{k}}}{\partial t} + \mathrm{i}\mathbf{k}\cdot\boldsymbol{\Omega} f_{\mathbf{k}} - \mathrm{i}\mathbf{k}\cdot\nabla_{\mathbf{J}} f_0\Phi_{\mathrm{s}\mathbf{k}}\right). \tag{6.13}$$

The integrand comprises three terms. The middle term is proportional to $\mathbf{k}\cdot\boldsymbol{\Omega}|f_{\mathbf{k}}|^2$ and yields zero because the sum contains both $\mathbf{k}$ and $-\mathbf{k}$. Given that $f^*_{\mathbf{k}} = f_{-\mathbf{k}}$, the integral over the first term is

$$-\tfrac{1}{2}(2\pi)^3\int \mathrm{d}^3\mathbf{J}\sum_{\mathbf{k}}\frac{\mathbf{k}\cdot\boldsymbol{\Omega}}{\mathbf{k}\cdot\nabla_{\mathbf{J}} f_0}\frac{\partial|f_{\mathbf{k}}|^2}{\partial t} = \frac{\mathrm{d}K}{\mathrm{d}t}, \tag{6.14a}$$

where

$$K \equiv -4\pi^3\int \mathrm{d}^3\mathbf{J}\sum_{\mathbf{k}}\frac{\mathbf{k}\cdot\boldsymbol{\Omega}}{\mathbf{k}\cdot\nabla_{\mathbf{J}} f_0}|f_{\mathbf{k}}|^2 \tag{6.14b}$$

will prove to be the disturbance's kinetic energy.

In preparation for handling the third term, note that

$$\int \mathrm{d}^3\boldsymbol{\theta}\, f_1[\Phi_\mathrm{s}, H_0] = \int \mathrm{d}^3\boldsymbol{\theta}\, f_1 \frac{\partial \Phi_\mathrm{s}}{\partial \boldsymbol{\theta}} \cdot \boldsymbol{\Omega} = (2\pi)^3 \sum_{\mathbf{k}} f^*_{\mathbf{k}} \mathrm{i}\mathbf{k}\cdot\boldsymbol{\Omega}\Phi_{\mathbf{sk}}. \tag{6.15}$$

Hence the third term in equation (6.13) is

$$\begin{aligned} \frac{\mathrm{d}P}{\mathrm{d}t} &\equiv (2\pi)^3 \int \mathrm{d}^3\mathbf{J} \sum_{\mathbf{k}} \mathrm{i}\mathbf{k}\cdot\boldsymbol{\Omega}\Phi_{\mathbf{sk}} f^*_{\mathbf{k}} = \int \mathrm{d}^6\mathbf{w}\, f_1[\Phi_\mathrm{s}, H_0] \\ &= \int \mathrm{d}^6\mathbf{w}\, f_1 \mathbf{v}\cdot\nabla\Phi_\mathrm{s}. \end{aligned} \tag{6.16}$$

Now we use integration by parts to move ∇ from Φ_s to $f_1\mathbf{v}$. Then we execute the integral over $\mathbf{v}$ to obtain $\rho_1\langle\mathbf{v}\rangle$ and finally we use the continuity equation $\partial\rho/\partial t = -\nabla\cdot(\rho\,\langle\mathbf{v}\rangle)$ to obtain

$$\frac{\mathrm{d}P}{\mathrm{d}t} = \int \mathrm{d}^3\mathbf{x}\,\Phi_\mathrm{s}\frac{\partial\rho_1}{\partial t} = -\frac{\mathrm{d}}{\mathrm{d}t}\tfrac{1}{2}G\int \mathrm{d}^3\mathbf{x}\,\mathrm{d}^3\mathbf{x}'\frac{\rho_1(\mathbf{x})\rho_1(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}, \tag{6.17}$$

Thus the two non-zero terms on the right of equation (6.13) are total time derivatives, so

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \frac{\mathrm{d}K}{\mathrm{d}t} + \frac{\mathrm{d}P}{\mathrm{d}t} \tag{6.18}$$

and it follows that the energy of the disturbance is

$$\begin{aligned} E &= K + P \\ &= -4\pi^3 \int \mathrm{d}^3\mathbf{J} \sum_{\mathbf{k}} \frac{\mathbf{k}\cdot\boldsymbol{\Omega}}{\mathbf{k}\cdot\nabla_{\mathbf{J}} f_0}|f_{\mathbf{k}}|^2 - \tfrac{1}{2}G\int \mathrm{d}^3\mathbf{x}\mathrm{d}^3\mathbf{x}'\frac{\rho_1(\mathbf{x})\rho_1(\mathbf{x}')^2}{|\mathbf{x}-\mathbf{x}'|}. \end{aligned} \tag{6.19}$$

Since P is manifestly the potential energy of the disturbance, K must be its kinetic energy as we stated above. When equation (6.5) is used to eliminate $|\mathbf{x}-\mathbf{x}'|$, we obtain an alternative expression for E

EofDF

$$E = -4\pi^3 \int \mathrm{d}^3\mathbf{J} \sum_{\mathbf{k}} \frac{\mathbf{k}\cdot\boldsymbol{\Omega}}{\mathbf{k}\cdot\nabla_{\mathbf{J}} f_0}|f_{\mathbf{k}}|^2 - \tfrac{1}{2}\mathcal{E}\sum_{\alpha}|A_\alpha|^2. \tag{6.20}$$

The ratio $\mathbf{k}\cdot\boldsymbol{\Omega}/\mathbf{k}\cdot\nabla_{\mathbf{J}} f_0$ that multiplies $|f_{\mathbf{k}}|^2$ in our expression for the kinetic energy is normally negative because when $\mathbf{k}$ points in the direction of increasing H_0 and $\mathbf{k}\cdot\boldsymbol{\Omega} > 0$, we expect this to be a direction in which f_0 decreases so $\mathbf{k}\cdot\nabla_{\mathbf{J}} f_0 < 0$. Hence the kinetic energy is expected to be positive while the potential energy is clearly negative.

NMsec

6.3 Normal modes

Normal modes

Linear disturbances of an equilibrium system always have solutions with time-dependence $\mathrm{e}^{-\mathrm{i}\omega t}$. We call such solutions normal modes, and they constitute some of the most powerful tools of physics. When we substitute

$$f_{\mathbf{k}}(\mathbf{J},t)=f_{\mathbf{k}}(\mathbf{J})\mathrm{e}^{-\mathrm{i}\omega t}\quad;\quad \Phi_{\mathbf{k}}(\mathbf{J},t)=\Phi_{\mathbf{k}}(\mathbf{J},\omega)\mathrm{e}^{-\mathrm{i}\omega t}$$

into equation (6.1) it becomes

prevK

$$(\mathbf{k}\cdot\boldsymbol{\Omega}-\omega)f_{\mathbf{k}}=(\mathbf{k}\cdot\nabla_{\mathbf{J}}f_0)\Phi_{\mathbf{k}}. \tag{6.21}$$

Since $\Phi_{\mathbf{k}}$ is a linear functional of the $f_{\mathbf{k}}$, this equation bears a strong resemblance to $\mathbf{A}\cdot\mathbf{b}=0$, with the vector $\mathbf{b}$ playing the role of $f_{\mathbf{k}}$ and $\mathbf{A}$ a matrix. The matrix equation has non-trivial solutions only if $|\mathbf{A}|=0$ and one might expect the same principle to require ω to take particular (eigen)values. The analogy is defective, however, because $\Phi_{\mathbf{k}}$ depends only infinitesimally on any of the $f_{\mathbf{k}}$: its computation involves integrals over $\mathbf{J}$ as well as sums over $\mathbf{k}$. Consequently, its value is essentially independent of the value of any individual $f_{\mathbf{k}}(\mathbf{J})$ and can be regarded as a given. This being so, equation (6.21) specifies $f_{\mathbf{k}}(\mathbf{J})$ at any value of $\mathbf{J}$ for which $\mathbf{k}\cdot\boldsymbol{\Omega}\neq\omega$, but provides no information regarding $f_{\mathbf{k}}(\mathbf{J})$ at the small number of actions for which $\mathbf{k}\cdot\boldsymbol{\Omega}=\omega$. Therefore we should write the solution to equation (6.21) as

DFvK

$$f_{\mathbf{k}}(\mathbf{J},\omega)=\mathcal{P}\frac{\mathbf{k}\cdot\nabla_{\mathbf{J}}f_0}{\mathbf{k}\cdot\boldsymbol{\Omega}-\omega}\Phi_{\mathbf{k}}+g_{\mathbf{k}}(\mathbf{J},\omega)\delta(\mathbf{k}\cdot\boldsymbol{\Omega}-\omega), \tag{6.22}$$

where $g_{\mathbf{k}}$ is a function to be determined and the symbol $\mathcal{P}$ implies that points at which $\mathbf{k}\cdot\boldsymbol{\Omega}=\omega$ are excluded. The Dirac delta plugs the gaps left by the $\mathcal{P}$. The equation coherently specifies that $f_{\mathbf{k}}(\mathbf{J})$ grows without limit as $\mathbf{k}\cdot\boldsymbol{\Omega}(\mathbf{J})\to\omega$ and becomes infinite when equality is reached. The presence of a Dirac delta in a DF is not problematic because measurable predictions involve integrals over a finite range of actions, and as such are free of Dirac deltas.

Multiplying equation (6.22) by $\mathrm{d}^6\mathbf{w}\,\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\boldsymbol{\theta}}\Phi^{(\alpha')*}$ and integrating over phase space, the left side becomes

DFvKtimes

$$\int\mathrm{d}^6\mathbf{w}\,\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\boldsymbol{\theta}}\Phi^{(\alpha')*}f_{\mathbf{k}}=(2\pi)^3\int\mathrm{d}^3\mathbf{J}\,\Phi_{\mathbf{k}}^{(\alpha')}f_{\mathbf{k}}. \tag{6.23a}$$

In the first term on the right of equation (6.22) we replace $\Phi_{\mathbf{k}}$ by $\sum_\alpha A_\alpha\Phi_{\mathbf{k}}^{(\alpha)}$ and then this first term yields

$$\begin{aligned}\mathcal{P}\int\mathrm{d}^3\mathbf{J}\,\frac{\mathbf{k}\cdot\nabla_{\mathbf{J}}f_0}{\mathbf{k}\cdot\boldsymbol{\Omega}-\omega}\sum_\alpha A_\alpha\Phi_{\mathbf{k}}^{(\alpha)}\int\mathrm{d}^3\boldsymbol{\theta}\,\Phi^{(\alpha')*}\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\boldsymbol{\theta}}\\ =(2\pi)^3\mathcal{P}\int\mathrm{d}^3\mathbf{J}\,\frac{\mathbf{k}\cdot\nabla_{\mathbf{J}}f_0}{\mathbf{k}\cdot\boldsymbol{\Omega}-\omega}\sum_\alpha A_\alpha\Phi_{\mathbf{k}}^{(\alpha)}\Phi_{\mathbf{k}}^{(\alpha')*}.\end{aligned} \tag{6.23b}$$

The last term on the right yields

$$(2\pi)^3 \int \mathrm{d}^3\mathbf{J}\, \Phi_{\mathbf{k}}^{(\alpha')*} g_{\mathbf{k}} \delta(\mathbf{k}\cdot\mathbf{\Omega} - \omega). \tag{6.23c}$$

Gathering the three terms together, equation (6.22) yields

$$\begin{aligned}(2\pi)^3 \int \mathrm{d}^3\mathbf{J}\, \Phi_{\mathbf{k}}^{(\alpha')} f_{\mathbf{k}} = (2\pi)^3 \mathcal{P} \int \mathrm{d}^3\mathbf{J}\, \frac{\mathbf{k}\cdot\nabla_{\mathbf{J}} f_0}{\mathbf{k}\cdot\mathbf{\Omega} - \omega} \sum_\alpha A_\alpha \Phi_{\mathbf{k}}^{(\alpha)} \Phi_{\mathbf{k}}^{(\alpha')*} \\ + (2\pi)^3 \int \mathrm{d}^3\mathbf{J}\, \Phi_{\mathbf{k}}^{(\alpha')*} g_{\mathbf{k}} \delta(\mathbf{k}\cdot\mathbf{\Omega} - \omega).\end{aligned} \tag{6.24}$$

When we sum both sides of this equation over $\mathbf{k}$, the left side becomes $-\mathcal{E} A_{\alpha'}$ by equation (6.6), and the first term on the right can be written $-\mathcal{E}\sum_\alpha P_{\alpha'\alpha} A_\alpha$ where $\mathbf{P}$ is the **polarisation operator** polarisation operator!

defPmatrix

$$P_{\alpha'\alpha}(\omega) \equiv -\frac{(2\pi)^3}{\mathcal{E}} \mathcal{P} \int \mathrm{d}^3\mathbf{J} \sum_{\mathbf{k}} \frac{\mathbf{k}\cdot\nabla_{\mathbf{J}} f_0}{\mathbf{k}\cdot\mathbf{\Omega} - \omega} \Phi_{\mathbf{k}}^{(\alpha)} \Phi_{\mathbf{k}}^{(\alpha')*}. \tag{6.25}$$

Moving this to the left side to gather terms proportional to A_α, the equation takes the form

vKmatrix

$$\sum_\alpha M_{\alpha'\alpha} A_\alpha = B_{\alpha'}, \tag{6.26a}$$

where

$$M_{\alpha'\alpha} \equiv \delta_{\alpha'\alpha} - P_{\alpha'\alpha} \quad ; \quad B_{\alpha'} \equiv -\frac{(2\pi)^3}{\mathcal{E}} \int \mathrm{d}^3\mathbf{J} \sum_{\mathbf{k}} g_{\mathbf{k}}(\mathbf{J}) \Phi_{\mathbf{k}}^{(\alpha')*} \delta(\mathbf{k}\cdot\mathbf{\Omega} - \omega). \tag{6.26b}$$

The **response matrix** $\mathbf{M}$ is completely specified by the unperturbed system, the set of potential-density pairs and the frequency ω. We expect $\mathbf{M}$ to have an inverse for almost any real ω, so equation (6.26a) can be solved for the A_α (which specify the mode's real-space structure) in terms of the B_α. Hence there is a normal mode of the system for any frequency that is equal to $\mathbf{k}\cdot\mathbf{\Omega}$ for some values $\mathbf{k}$ and $\mathbf{J}$. This situation differs dramatically from the model set by normal mechanical systems such as bells. Their normal mode frequencies are set by equations analogous to $\mathbf{M}\cdot\mathbf{A} = 0$ that force $\mathbf{M}$ to be degenerate. Here $\mathbf{M}\cdot\mathbf{A}$ is allowed to be non-zero, so the normal-mode frequencies are largely unconstrained. response matrix!

Actually, there are infinitely many normal modes at a given frequency, because we can solve for $\mathbf{A}$ given any $\mathbf{B}$ and the latter is determined by the unconstrained function $g_{\mathbf{k}}(\mathbf{J})$. The situation is analogous to perturbation of a quantum Hamiltonian with a degenerate eigenvalue except that in that case the space spanned by the degenerate stationary states generally has a finite dimension.

dressSec

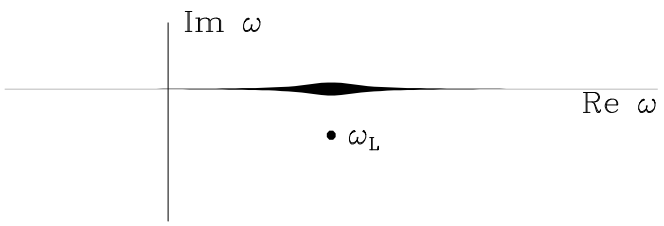

Figure 6.1 Modes near a singular frequency are heavily dressed – the width of the real axis symbolises the dressing ($\mathbf{M}^{-1}-\mathbf{I}$) of the mode of frequency ω, which diminishes with distance from the pole of $\mathbf{M}^{-1}$ at $\omega_{\rm L}$.

dresserfig

6.3.1 Particle dressing

Particle dressing

The coefficients B_α prove to be the expansion coefficients of the density

$$\rho_B(\mathbf{x}) = \int \mathrm{d}^3\mathbf{v} \sum_{\mathbf{k}} g_{\mathbf{k}}(\mathbf{J})\delta(\mathbf{k}\cdot\mathbf{\Omega}-\omega) \tag{6.27}$$

generated by the DF $\sum_{\mathbf{k}} g_{\mathbf{k}}\delta(\mathbf{k}\cdot\mathbf{\Omega}-\omega)$. Indeed

$$\begin{aligned}\int \mathrm{d}^3\mathbf{x}\,\Phi^{(\alpha)*}\rho_B &= \int \mathrm{d}^3\mathbf{x}\,\Phi^{(\alpha)*}\sum_{\mathbf{k}}\int \mathrm{d}^3\mathbf{v}\, g_{\mathbf{k}}(\mathbf{J})\delta(\mathbf{k}\cdot\mathbf{\Omega}-\omega)\\ &= \sum_{\mathbf{k}\mathbf{k}'}\int \mathrm{d}^3\mathbf{J}\int \mathrm{d}^3\boldsymbol{\theta}\,\Phi^{(\alpha)*}_{\mathbf{k}'}\mathrm{e}^{-\mathrm{i}\mathbf{k}'\cdot\boldsymbol{\theta}}g_{\mathbf{k}}\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\boldsymbol{\theta}}\delta(\mathbf{k}\cdot\mathbf{\Omega}-\omega)\\ &= -\mathcal{E}B_\alpha.\end{aligned} \tag{6.28}$$

The coefficients A_α are the expansion coefficients of the mode's density. In the absence of self gravity, the matrix $\mathbf{P}$ would vanish and $\mathbf{M}$ would reduce to the identity, so A_α would equal B_α. In the next chapter we will explore a toy model, the periodic cube, in which it is obvious that $g_{\mathbf{k}}(\mathbf{J})$ can be varied without changing the B_α, so it is likely that modes are not uniquely specified by their footprints $(\mathbf{A},\mathbf{B})$ in real space.

Physically, in the absence of self gravity, a non-uniform distribution of stars on a resonant torus $\mathbf{J}$ would create the density distribution described by $\mathbf{B}$, which would oscillate at frequency ω for ever. In the presence of self gravity, the oscillating potential created by the resonant stars drives sympathetic oscillations of non-resonant stars. $\mathbf{A}$ quantifies the density distribution formed by the resonant and non-resonant stars together, so $\mathbf{A}-\mathbf{B}=(\mathbf{M}^{-1}-\mathbf{I})\cdot\mathbf{B}$. We say that the **bare distribution** described by $\mathbf{B}$ produces the **dressed** distribution described by $\mathbf{A}$.

bare distribution!

dressed!

The extent to which self gravity dresses a bare distribution grows without limit as $\mathbf{M}$ tends to a singular matrix. It won't be singular for any real frequency, but it will be for some complex frequencies $\omega_{\rm L}$, and if one of these lies close to the real line, the modes that have frequencies on the closest section of the real line will be heavily dressed (Figure 6.1). The closer to the real line $\omega_{\rm L}$ lies, the shorter is the heavily dressed section of the line, so the narrower is the range $\Delta\omega$ of frequencies of the heavily dressed normal modes that will tend to dominate disturbances.

contspecSec

Implication of continuous spectra

6.3.2 Implication of continuous spectra

When a system's spectrum forms a continuum, we should not expect any one normal mode to be excited: no matter how delicately we excite the system, we will excite an infinite number of normal modes. Indeed, the expression for the system's energy will take the form of an integral $E = \int \mathrm{d}\omega |a_\omega|^2 \mathcal{E}_\omega$, where a_ω is the amplitude of the mode and $\mathcal{E}_\omega$ is the modal energy density at unit amplitude, so the energy in any one mode will be infinitesimal. When we do excite modes that cover a range in frequencies of width $\Delta\omega$, the phases of the excited modes will drift with respect to one another such that if they were identical at $t = 0$, by time t they will cover a range $\Delta\phi \sim t\Delta\omega$. Once this range becomes comparable to 2π, the contributions of modes to measurables will tend to cancel and the disturbance will seem to have died away. Each mode will, however, have the same amplitude and energy as it did originally, but collective effects will damp on a timescale $\sim 2\pi/\Delta\omega$. We saw above that when a pole at $\omega_{\rm L}$ of $\mathbf{M}^{-1}$ lies close to the real axis, the frequency range $\Delta\omega$ of heavily dressed modes is narrow. Since these modes will dominate measureables, small values of $\Im\mathrm{m}\,\omega_{\rm L}$ are associated with long-lived ('weakly damped') disturbances.

Much of the utility of normal modes derives from three properties:

- any initial condition can be expressed as a linear combination of normal modes;
- a normal mode's time evolution is trivial: multiply the amplitude by $\mathrm{e}^{-\mathrm{i}\omega t}$;
- the energies of normal modes are additive.

The first two properties make normal modes powerful tools for the solution of initial-value problems.

It's instructive to compare the normal modes of a gas ball such as the Sun with those of a globular cluster. The Sun's normal modes have discrete frequencies, that range (in principle) from a minimum frequency up to infinity. Hence the dynamical state of the Sun can be specified by either (i) giving the velocity $\mathbf{v}(\mathbf{x})$ and temperature $T(\mathbf{x})$ of the gas at $\mathbf{x}$, or (ii) giving the (complex) amplitudes of a countable infinity of normal modes. To see that the information content of the two possible specifications are equal, one just needs to expand $T(\mathbf{x})$ and each component of $\mathbf{v}(\mathbf{x})$ in a set of basis functions such as $f_{n\ell}(r)Y_\ell^m(\vartheta,\phi)$.

We only need functions on real space to specify the dynamical state of the Sun because we know that the atomic velocity distribution at any point is Maxwellian, and is thus specified by its mean and dispersion, i.e., $\mathbf{v}$ and T. The velocity distribution in a globular cluster, by contrast, is never Maxwellian. Hence vastly more information is required to specify the dynamical state of a globular cluster than the state of the Sun, and the amplitudes of a countable infinity of normal modes don't contain sufficient information: we need the amplitudes of an uncountable infinity of normal modes.

Given the state $\mathcal{F}(\boldsymbol{\theta},\mathbf{J})$ of f_1 at $t = 0$, it would be convenient to write this as a linear combination of normal modes, for then we could predict f_1 at arbitrary times by multiplying each mode DF by $\mathrm{e}^{-\mathrm{i}\omega t}$. Equation (6.22)

implies that the task of decomposition into normal modes amounts to finding functions $g_{\mathbf{k}}(\mathbf{J}|\omega)$ such that

$$\mathcal{F}_{\mathbf{k}}(\boldsymbol{\theta},\mathbf{J}) = \int \mathrm{d}\omega\, \mathcal{P}\frac{\mathbf{k}\cdot\nabla_{\mathbf{J}} f_0}{\mathbf{k}\cdot\boldsymbol{\Omega}-\omega}\Phi_{\mathbf{k}} + g_{\mathbf{k}}(\mathbf{J}|\mathbf{k}\cdot\boldsymbol{\Omega}), \tag{6.29}$$

where $\Phi_{\mathbf{k}} = \sum_\alpha A_\alpha \Phi_{\mathbf{k}}^{(\alpha)}$ with $\mathbf{A}$ determined by the $g_{\mathbf{k}}$ via equations (6.26).

6.3.3 Orthogonality of normal modes

Orthogonality of normal modes

Before we can talk about orthogonality, we need to specify an inner product on the space of disturbed DFs. We define the inner product of two DFs $f(\boldsymbol{\theta},\mathbf{J})$ and $\widetilde{f}(\boldsymbol{\theta},\mathbf{J})$ to be

innerprod

$$\langle f|\widetilde{f}\rangle = -4\pi^3 \int \mathrm{d}^3\mathbf{J} \sum_{\mathbf{k}} \frac{\mathbf{k}\cdot\boldsymbol{\Omega}}{\mathbf{k}\cdot\nabla_{\mathbf{J}} f_0} f_{\mathbf{k}}^* \widetilde{f}_{\mathbf{k}} - \tfrac{1}{2}\mathcal{E}\sum_\alpha A_\alpha^* \widetilde{A}_\alpha, \tag{6.30}$$

where A_α describes the spatial structure implied by f and $\widetilde{A}_\alpha$ does the same for $\widetilde{f}$. This definition is such that the energy (6.20) of a disturbance is just the modulus of f:

$$E = \langle f|f\rangle. \tag{6.31}$$

The condition for the energies of modes to be additive, is that modes are mutually orthogonal under this inner product: if two DFs f and g are expanded in normal modes, $f(\mathbf{w}) = \int \mathrm{d}\omega\, a(\omega) f_\omega(\mathbf{w})$ and $g(\mathbf{w}) = \int \mathrm{d}\omega\, b(\omega) f_\omega(\mathbf{w})$, then

$$\begin{aligned}\langle g|f\rangle &= \int \mathrm{d}\omega \int \mathrm{d}\omega'\, a^*(\omega) b(\omega') \langle f_\omega | f_{\omega'}\rangle \\ &= \int \mathrm{d}\omega \int \mathrm{d}\omega'\, a^*(\omega) b(\omega') \delta(\omega-\omega') = \int \mathrm{d}\omega\, |a(\omega)|^2 E_\omega,\end{aligned} \tag{6.32}$$

where $E_\omega \equiv \langle f_\omega | f_\omega\rangle$. After inserting DFs of two normal modes from equation (6.22) into equation (6.30), a slightly technical computation yields [38]

modeEmode

$$\begin{aligned}\langle f|\widetilde{f}\rangle &= -4\pi^3\omega\delta(\omega-\widetilde{\omega}) \\ &\quad\times \int \mathrm{d}^3\mathbf{J} \sum_{\mathbf{k}} \left\{ \pi^2 (\mathbf{k}\cdot\nabla_{\mathbf{J}} f_0)\Phi_{\mathbf{k}}^* \widetilde{\Phi}_{\mathbf{k}} + \frac{g_{\mathbf{k}}^* \widetilde{g}_{\mathbf{k}}}{\mathbf{k}\cdot\nabla_{\mathbf{J}} f_0} \right\} \delta(\mathbf{k}\cdot\boldsymbol{\Omega}-\omega),\end{aligned} \tag{6.33}$$

where $\Phi_{\mathbf{k}}$ are the Fourier components of the potential that f generates and similarly for $\widetilde{\Phi}_{\mathbf{k}}$. Equation (6.33) establishes that modes with different frequencies are indeed orthogonal. As remarked above, there should be more than one normal mode at each frequency, and the challenge posed by this remark is to find functions $g_{\mathbf{k}}$ that through equations (6.26) generate potentials $\Phi_{\mathbf{k}}$ that are mutually orthogonal in the sense of equation (6.33).

ergodicModes

6.3.4 Modes of ergodic systems

Modes of ergodic systems

For the case of an ergodic system, Antonov [39] constructed a second-order Hermitian operator $K_{\rm A}$ that has the normal modes for its eigenfunctions and the squares of their frequencies for its eigenvalues. By the reality of the eigenvalues of Hermitian operators, it follows that in the ergodic case ω^2 is real, so ω is either real or pure imaginary. Moreover, if ω is a normal-mode frequency, then so is $-\omega$ and a pure imaginary value of ω will be associated with both a decaying and a growing mode. Since $\mathbf{k}\cdot\mathbf{\Omega}$ is inherently real, the term in equation (6.22) containing $g_{\mathbf{k}}$ can be dropped when ω is imaginary. As a consequence, any imaginary normal-mode frequency is also one of the 'Landau frequencies' to be discussed below.

Likely to apply to t-reversible systems?

reference Polyachenko?

The energy of an exponentially decaying or growing mode must be zero since its amplitude could not otherwise change at constant energy. Antonov derived a criterion for the energies of all modes to be positive, which establishes stability. These results can be explored in Problems 6.1 to 6.3 and in [40].

6.3.5 Excitation of modes by Poisson noise

Excitation of modes by Poisson noise

When we randomly sample N coordinates from an analytic DF $f_0(\mathbf{J})$ of a system of mass M, we create the **Klimontovich DF**

Klimontovich DF!

KlimontovicDF
$$f_N(\boldsymbol{\theta},\mathbf{J}) = m\sum_{i=1}^{N}\delta(\boldsymbol{\theta}-\boldsymbol{\theta}_i)\delta(\mathbf{J}-\mathbf{J}_i) \qquad (m\equiv M/N), \tag{6.34}$$

which differs from f_0, so the realisation includes the perturbation

KlimontovicFone
$$f_1(\boldsymbol{\theta},\mathbf{J}) = f_N(\boldsymbol{\theta},\mathbf{J}) - f_0(\mathbf{J}). \tag{6.35}$$

The expectation value of f_1 vanishes but its square has a non-vanishing expectation value:

Klimfone
$$\begin{aligned}\big\langle f_1(\mathbf{J},\boldsymbol{\theta})f_1(\boldsymbol{\theta}',\mathbf{J}')\big\rangle &= \big\langle \big(f_N(\boldsymbol{\theta},\mathbf{J})-f_0(\mathbf{J})\big)\big(f_N(\boldsymbol{\theta}',\mathbf{J}')-f_0(\mathbf{J}')\big)\big\rangle \\ &= \big\langle f_N(\boldsymbol{\theta},\mathbf{J})f_N(\boldsymbol{\theta}',\mathbf{J}')\big\rangle - f_0(\mathbf{J})f_0(\mathbf{J}').\end{aligned} \tag{6.36}$$

After substituting from equation (6.34) for f_N, we need to distinguish between the cases in which $(\boldsymbol{\theta},\mathbf{J})$ and $(\boldsymbol{\theta}',\mathbf{J}')$ do or do not refer to the same particle.

$$\begin{aligned}\big\langle f_N(\boldsymbol{\theta},\mathbf{J})f_N(\boldsymbol{\theta}',\mathbf{J}')\big\rangle &= m^2\sum_{i\neq j}\big\langle\delta(\boldsymbol{\theta}-\boldsymbol{\theta}_i)\delta(\mathbf{J}-\mathbf{J}_i)\delta(\boldsymbol{\theta}'-\boldsymbol{\theta}_j)\delta(\mathbf{J}'-\mathbf{J}_j)\big\rangle \\ &\quad + m^2\sum_{i}\big\langle\delta(\boldsymbol{\theta}-\boldsymbol{\theta}_i)\delta(\mathbf{J}-\mathbf{J}_i)\delta(\boldsymbol{\theta}'-\boldsymbol{\theta}_i)\delta(\mathbf{J}'-\mathbf{J}_i)\big\rangle \\ &= m^2\sum_{i\neq j}\langle\delta(\boldsymbol{\theta}-\boldsymbol{\theta}_i)\delta(\mathbf{J}-\mathbf{J}_i)\rangle\big\langle\delta(\boldsymbol{\theta}'-\boldsymbol{\theta}_j)\delta(\mathbf{J}'-\mathbf{J}_j)\big\rangle \\ &\quad + m^2\sum_{i}\langle\delta(\boldsymbol{\theta}-\boldsymbol{\theta}_i)\delta(\mathbf{J}-\mathbf{J}_i)\rangle\,\delta(\boldsymbol{\theta}'-\boldsymbol{\theta})\delta(\mathbf{J}'-\mathbf{J}),\end{aligned} \tag{6.37}$$

where the second equality follows because particles i and j are not correlated. Now

$$M\,\langle\delta(\boldsymbol{\theta}-\boldsymbol{\theta}_i)\delta(\mathbf{J}-\mathbf{J}_i)\rangle = \langle f_N(\boldsymbol{\theta},\mathbf{J})\rangle = f_0(\mathbf{J}) \tag{6.38}$$

so the autocorrelation of the Klimontovich DF is

$$\left\langle f_N(\boldsymbol{\theta},\mathbf{J})f_N(\boldsymbol{\theta}',\mathbf{J}')\right\rangle = \frac{N^2-N}{N^2}f_0(\mathbf{J})f_0(\mathbf{J}') + mf_0(\mathbf{J})\delta(\boldsymbol{\theta}'-\boldsymbol{\theta})\delta(\mathbf{J}'-\mathbf{J}). \tag{6.39}$$

When we use this relation in equation (6.36), we find

PoissonF
$$\left\langle f_1(\mathbf{J},\boldsymbol{\theta})f_1(\boldsymbol{\theta}',\mathbf{J}')\right\rangle = mf_0(\mathbf{J})\delta(\boldsymbol{\theta}'-\boldsymbol{\theta})\delta(\mathbf{J}'-\mathbf{J}) - \frac{1}{N}f_0(\mathbf{J})f_0(\mathbf{J}'). \tag{6.40}$$

We multiply by $\mathrm{e}^{\mathrm{i}(\mathbf{k}\cdot\boldsymbol{\theta}+\mathbf{k}'\boldsymbol{\theta}')}/(2\pi)^6$ and integrate over the angle variables to obtain the correlation of the Fourier amplitudes that arise from randomly sampling f_0

PoissonCF
$$\langle f_{\mathbf{k}}(\mathbf{J})f_{\mathbf{k}'}(\mathbf{J}')\rangle = \frac{mf_0(\mathbf{J})}{(2\pi)^3}\delta_{\mathbf{k},-\mathbf{k}'}\delta(\mathbf{J}-\mathbf{J}') \quad \mathbf{k}\neq 0. \tag{6.41}$$

We have seen that in the absence of self-gravity, the density distribution associated with a perturbation is fully described by the expansion coefficients $A_\alpha = B_\alpha$, the coefficients generated by the part of the DF that's specified by $g_{\mathbf{k}}(\mathbf{J})$. In this case, the perturbation to the DF is just that (6.35) generated by Poisson noise. Hence the DF that $g_{\mathbf{k}}(\mathbf{J})$ generates must yield the expectations values of equation (6.41):

gPoisson
$$\int \mathrm{d}\omega\,\langle g_{\mathbf{k}}(\mathbf{J},\omega)g_{\mathbf{k}'}(\mathbf{J}',\omega)\rangle\,\delta(\mathbf{k}\cdot\boldsymbol{\Omega}-\omega)\delta(\mathbf{k}'\cdot\boldsymbol{\Omega}'-\omega) = \frac{mf_0(\mathbf{J})}{(2\pi)^3}\delta_{\mathbf{k},-\mathbf{k}'}\delta(\mathbf{J}-\mathbf{J}'). \tag{6.42}$$

Below we will use this equation to compute diffusion coefficients.

Suppose an N-body simulation is initialised by randomly sampling a DF $f_0(\mathbf{J})$ and its particles are advanced by integrating their equations of motion in the mean-field potential defined by f_0. Then the autocorrelation function of f_1 remains that given by equation (6.40). This invariance signals that the phases of the DF's constituent normal modes were initially uniformly distributed on $(0, 2\pi)$ so nothing changes as the phases of the modes evolve at different rates.

When the particles are advanced in the potential that they mutually generate, the autocorrelation function evolves away from that given by equation (6.40) because the self-consistent system has different normal modes from the test-particle system, and when the initial DF is decomposed into these normal modes, the initial phases are not uniformly distributed. The autocorrelation evolves as the phases increase at different rates, gradually generating a uniform distribution of phases. During this process the simulation particles become dressed – in the initial conditions the dressing included in each normal mode interfered destructively with the dressing included in other normal modes by virtue of the modes having non-uniformly distributed phases. These cancellations diminish, and the dressing becomes manifest, as the phases of modes tend to a uniform distribution.

6.3.6 Diffusion coefficients

Diffusion coefficients

By equation (5.5), the diffusive flux $\mathbf{F}$ is driven by the expectations $\langle \Phi_{\mathbf{k}} f^*_{\mathbf{k}} \rangle$. We now calculate this for a single normal mode of frequency ω. Then we will obtain the total diffusion coefficients by adding the contributions of each frequency.

In view of equation (6.22)

whatwant

$$\begin{aligned}\langle \Phi_{\mathbf{k}} f^*_{\mathbf{k}} \rangle (\mathbf{J}) &= \left\langle \Phi_{\mathbf{k}}(\mathbf{J}) \Big(\mathcal{P} \frac{\mathbf{k}\cdot\nabla_{\mathbf{J}} f_0}{\mathbf{k}\cdot\boldsymbol{\Omega} - \omega} \Phi^*_{\mathbf{k}}(\mathbf{J}) + g^*_{\mathbf{k}}(\mathbf{J}) \delta(\mathbf{k}\cdot\boldsymbol{\Omega} - \omega) \Big) \right\rangle \\ &= \mathcal{P} \frac{\mathbf{k}\cdot\nabla_{\mathbf{J}} f_0}{\mathbf{k}\cdot\boldsymbol{\Omega} - \omega} \langle \Phi_{\mathbf{k}}(\mathbf{J}) \Phi^*_{\mathbf{k}}(\mathbf{J}) \rangle + \langle \Phi_{\mathbf{k}}(\mathbf{J}) g^*_{\mathbf{k}}(\mathbf{J}) \rangle \, \delta(\mathbf{k}\cdot\boldsymbol{\Omega} - \omega)\end{aligned} \tag{6.43}$$

Now

defsE

$$\begin{aligned}\Phi_{\mathbf{k}}(\mathbf{J},\omega) &= \sum_\alpha A_\alpha \Phi^{(\alpha)}_{\mathbf{k}}(\mathbf{J}) = \sum_{\alpha\alpha'} M^{-1}_{\alpha\alpha'}(\omega) B_{\alpha'} \Phi^{(\alpha)}_{\mathbf{k}}(\mathbf{J}) \\ &= -\frac{(2\pi)^3}{\mathcal{E}} \int \mathrm{d}^3\mathbf{J}' \sum_{\mathbf{k}'} \sum_{\alpha\alpha'} \Phi^{(\alpha)}_{\mathbf{k}}(\mathbf{J}) M^{-1}_{\alpha\alpha'} \Phi^{(\alpha')*}_{\mathbf{k}'}(\mathbf{J}') g_{\mathbf{k}'} \delta(\mathbf{k}'\cdot\boldsymbol{\Omega}' - \omega) \\ &= -(2\pi)^3 \int \mathrm{d}^3\mathbf{J}' \sum_{\mathbf{k}'} E_{\mathbf{k}\mathbf{k}'}(\mathbf{J},\mathbf{J}',\omega) g_{\mathbf{k}'}(\mathbf{J}') \delta(\mathbf{k}'\cdot\boldsymbol{\Omega}' - \omega),\end{aligned} \tag{6.44a}$$

where

$$E_{\mathbf{k}\mathbf{k}'}(\mathbf{J},\mathbf{J}',\omega) \equiv \frac{1}{\mathcal{E}} \sum_{\alpha\alpha'} \Phi^{(\alpha)}_{\mathbf{k}}(\mathbf{J}) M^{-1}_{\alpha\alpha'}(\omega) \Phi^{(\alpha')*}_{\mathbf{k}'}(\mathbf{J}') \tag{6.44b}$$

is angle-action representation of the matrix $\mathbf{M}^{-1}$. The rhs of equation (6.43) contains the expectation value

$$\langle \Phi_{\mathbf{k}} g^*_{\mathbf{k}} \rangle = -(2\pi)^3 \int \mathrm{d}^3\mathbf{J}' \sum_{\mathbf{k}'} E_{\mathbf{k}\mathbf{k}'}(\mathbf{J},\mathbf{J}',\omega) \, \langle g_{\mathbf{k}'}(\mathbf{J}') g^*_{\mathbf{k}}(\mathbf{J}) \rangle \, \delta(\mathbf{k}'\cdot\boldsymbol{\Omega}' - \omega).$$

Using equation (6.42) for $\langle g_{\mathbf{k}'}(\mathbf{J}') g^*_{\mathbf{k}''}(\mathbf{J}'') \rangle$, yields

$$\begin{aligned}\langle \Phi_{\mathbf{k}} g^*_{\mathbf{k}} \rangle &= -m \int \mathrm{d}^3\mathbf{J}' E_{\mathbf{k}\mathbf{k}}(\mathbf{J},\mathbf{J}',\omega) f_0(\mathbf{J}) \delta(\mathbf{J}-\mathbf{J}') \delta(\mathbf{k}\cdot\boldsymbol{\Omega}' - \omega) \\ &= -m E_{\mathbf{k}\mathbf{k}}(\mathbf{J},\mathbf{J},\omega) f_0(\mathbf{J}) \delta(\mathbf{k}\cdot\boldsymbol{\Omega} - \omega).\end{aligned} \tag{6.45}$$

Hence a contribution to the flux (6.43) is

finalFone

$$\begin{aligned}\mathbf{F}^{(1)}(\mathbf{J}) &= -\mathrm{i} \sum_{\mathbf{k}} \mathbf{k} \int \mathrm{d}\omega \, \langle \Phi_{\mathbf{k}} g_{-\mathbf{k}} \rangle \\ &= \mathrm{i}m \sum_{\mathbf{k}} \mathbf{k} \int \mathrm{d}\omega \, E_{\mathbf{k}\mathbf{k}}(\mathbf{J},\mathbf{J},\omega) f_0(\mathbf{J}) \delta(\mathbf{k}\cdot\boldsymbol{\Omega} - \omega) \\ &= \mathbf{D}^{(1)}(\mathbf{J}) f_0(\mathbf{J}),\end{aligned} \tag{6.46a}$$

where $\mathbf{D}^{(1)}$ is the first-order diffusion coefficient

$$\mathbf{D}^{(1)}(\mathbf{J}) = \mathrm{i}m \sum_{\mathbf{k}} \mathbf{k} E_{\mathbf{kk}}(\mathbf{J}, \mathbf{J}, \mathbf{k}\cdot\boldsymbol{\Omega}). \tag{6.46b}$$

The other expectation value on the rhs of equation (6.43) is

$$\begin{aligned}\langle \Phi_{\mathbf{k}} \Phi^*_{\mathbf{k}}\rangle &= (2\pi)^6 \int \mathrm{d}^3\mathbf{J}' \mathrm{d}^3\mathbf{J}'' \sum_{\mathbf{k}'\mathbf{k}''} E_{\mathbf{kk}'}(\mathbf{J}, \mathbf{J}', \omega) E^*_{\mathbf{kk}''}(\mathbf{J}, \mathbf{J}'', \omega) \\ &\qquad \times \langle g_{\mathbf{k}'}(\mathbf{J}') g^*_{\mathbf{k}''}(\mathbf{J}'')\rangle\, \delta(\mathbf{k}'\cdot\boldsymbol{\Omega}' - \omega)\delta(\mathbf{k}''\cdot\boldsymbol{\Omega}'' - \omega) \\ &= (2\pi)^3 \int \mathrm{d}^3\mathbf{J}' \sum_{\mathbf{k}'} E_{\mathbf{kk}'}(\mathbf{J}, \mathbf{J}', \omega) E^*_{\mathbf{kk}'}(\mathbf{J}, \mathbf{J}', \omega) m f_0(\mathbf{J}') \delta(\mathbf{k}'\cdot\boldsymbol{\Omega}' - \omega).\end{aligned} \tag{6.47}$$

Hence the other contribution to the flux in equation (6.43) is

finalFtwo

$$\begin{aligned}\mathbf{F}^{(2)}(\mathbf{J}) &= -\mathrm{i}\sum_{\mathbf{k}} \mathbf{k} \int \mathrm{d}\omega\, \mathcal{P} \frac{\mathbf{k}\cdot\nabla_{\mathbf{J}} f_0}{\mathbf{k}\cdot\boldsymbol{\Omega} - \omega} \langle \Phi_{\mathbf{k}} \Phi^*_{\mathbf{k}}\rangle \\ &= -\mathrm{i}(2\pi)^3 m \sum_{\mathbf{k}} \mathbf{k} \int \mathrm{d}\omega\, \mathcal{P} \frac{\mathbf{k}\cdot\nabla_{\mathbf{J}} f_0}{\mathbf{k}\cdot\boldsymbol{\Omega} - \omega} \\ &\qquad \times \int \mathrm{d}^3\mathbf{J}' \sum_{\mathbf{k}'} |E_{\mathbf{kk}'}(\mathbf{J}, \mathbf{J}', \omega)|^2 f_0(\mathbf{J}') \delta(\mathbf{k}'\cdot\boldsymbol{\Omega}' - \omega) \\ &= \mathbf{D}^{(2)} \cdot \nabla_{\mathbf{J}} f_0,\end{aligned} \tag{6.48a}$$

where $\mathbf{D}^{(2)}$ is the second-order diffusion coefficient

$$\mathbf{D}^{(2)}(\mathbf{J}) = -\mathrm{i}(2\pi)^3 m \sum_{\mathbf{k}} \mathbf{k}\otimes\mathbf{k} \int \mathrm{d}^3\mathbf{J}' \sum_{\mathbf{k}'} \mathcal{P} \frac{|E_{\mathbf{kk}'}(\mathbf{J}, \mathbf{J}', \mathbf{k}'\cdot\boldsymbol{\Omega}')|^2}{\mathbf{k}\cdot\boldsymbol{\Omega} - \mathbf{k}'\cdot\boldsymbol{\Omega}'} f_0(\mathbf{J}'). \tag{6.48b}$$

This is a puzzling equation because it states that the absolutely real left side is equal to i times an apparently real expression. We need to show that the integral over real action space of an evidently real integrand is pure imaginary!

The problem is that we have treated stars as undamped oscillations, an assumption that is inconsistent with our expectation that the actions (and therefore energies) of these oscillators change. We get insight into the problem by considering a driven damped harmonic oscillator. Box 6.1 shows that the rate at which a driving force works on a damped oscillator tends to a non-zero limit as the damping term in the equation of motion tends to zero: the limit of damped behaviour is not equal to behaviour in the absence of damping.[1] Physically the issue comes down to the phase of the driving force relative to the oscillator's velocity $\dot{y}$: let the driving force $\cos\omega t$ produce a response $y = A\cos(\omega t + \phi)$, then the force works at rate

$$W = \dot{y}\cos\omega t = -A\omega \sin(\omega t + \phi)\cos(\omega t) = -\tfrac{1}{2}A\omega\big[\sin(2\omega t + \phi) - \sin\phi\big].$$

[1] In the same way, fluid flow at zero viscosity (inviscid flow) differs profoundly from flow in the limit of vanishing viscosity (flow at high Reynolds number).

Box 6.1: A damped oscillator

A damped oscillator has equation of motion $\ddot{y} + k\dot{y} + \omega_0^2 y = \mathrm{e}^{\mathrm{i}\omega t}$. We set $y = \mathrm{e}^{\alpha t}$ and conclude that the frequencies of its modes are

$$\alpha_\pm \equiv \mathrm{i}k/2 \pm \sqrt{\omega_0^2 - (k/2)^2} \quad ; \quad (\alpha_\pm^* = -\alpha_\mp). \tag{1}$$

To get the **particular integral**, we take $y = A\mathrm{e}^{\mathrm{i}\omega t}$ and obtain particular integral!

$$A = \frac{1}{\omega_0^2 - \omega^2 + \mathrm{i}k\omega} = \frac{-1}{(\omega - \alpha_+)(\omega - \alpha_-)}. \tag{2}$$

The rate at which the driving force works is $W = \dot{y}\cos\omega t$ and taking y to be given by the particular integral this is $-\omega\Im\mathrm{m}\,(y)\cos\omega t$.

$$\begin{aligned} W(t) &= \omega\Im\mathrm{m}\left(\frac{(\omega-\alpha_+^*)(\omega-\alpha_-^*)\mathrm{e}^{\mathrm{i}\omega t}}{|\omega-\alpha_+|^2|\omega-\alpha_-|^2}\right)\cos\omega t \\ &= \omega\Im\mathrm{m}\left(\frac{(\omega+\alpha_-)(\omega+\alpha_+)\mathrm{e}^{\mathrm{i}\omega t}}{|\omega-\alpha_+|^2|\omega-\alpha_-|^2}\right)\cos\omega t, \end{aligned} \tag{3}$$

where we have used $\alpha_+* = -\alpha_-$, etc. When we time-average $W(t)$, the term proportional to $\sin\omega t\cos\omega t$ drops out and we obtain

$$\overline{W} = \tfrac{1}{2}\omega^2 \frac{\Im\mathrm{m}\,(\alpha_+ + \alpha_-)}{|\omega-\alpha_+|^2|\omega-\alpha_-|^2} = \frac{\omega^2 k/2}{|\omega-\alpha_+|^2|\omega-\alpha_-|^2}. \tag{4}$$

Now we suppose that the driving force is uniformly spread over a continuum of frequencies and we compute the aggregate rate of working by integrating over frequencies. We convert the integral over real frequencies into a closed contour by adding a semi-circle above the real axis. The included poles are at $\omega = \alpha_\pm$ with residues

$$R_+ = \frac{\alpha_+^2 k/2}{(\alpha_+ - \alpha_+^*)|\alpha_+ - \alpha_-|^2} = \frac{\alpha_+^2 k/2}{(\alpha_+ + \alpha_-)4(\omega_0^2 - k^2/4)} = \frac{\alpha_+^2}{8\mathrm{i}(\omega_0^2 - k^2/4)}$$

$$R_- = \frac{\alpha_-^2 k/2}{|\alpha_- - \alpha_+|^2(\alpha_- - \alpha_-^*)} = \frac{\alpha_-^2}{8\mathrm{i}(\omega_0^2 - k^2/4)}.$$

Multiplying the sum of the residues by $2\pi\mathrm{i}$ we find that the integral is

$$\int_{-\infty}^{\infty} \mathrm{d}\omega\, \overline{W} = \frac{\pi(\alpha_+^2 + \alpha_-^2)}{4(\omega_0^2 - k^2/4)} = \frac{\pi(\frac{1}{2}k^2 + 2(\omega_0^2 - (k/2)^2))}{4(\omega_0^2 - k^2/4)} = \frac{\pi\omega_0^2}{2(\omega_0^2 - k^2/4)} \tag{5}$$

which tends to $\pi/2$ as $k \to 0$.

This averages to zero when $\phi = 0$ or π, as it does in the absence of damping.

The integral in equation (6.48b) describes the effect on an oscillator of frequency $\mathbf{k}\cdot\mathbf{\Omega}$ of being driven by a series of oscillators with frequencies $\mathbf{k}'\cdot\mathbf{\Omega}'$. The integrand changes sign suddenly each time the driving frequency passes through the frequency of the driven oscillator because the driven oscillator

is assumed to be undamped. If there is even the tiniest amount of damping, the phase difference ϕ between the driven and driving oscillators will swing continuously from zero to π (or vice-versa) and energy will be exchanged as ϕ passes through $\pi/2$. Box 6.1 shows that the mean rate of energy exchange is independent of the damping rate in the limit of small damping.

So we endow the driven oscillator of equation (6.48b) with damping by giving $\mathbf{k}\cdot\mathbf{\Omega}$ an infinitesimal, positive imaginary part. This shifts the integrand's pole at $\mathbf{k}\cdot\mathbf{\Omega}=\mathbf{k}'\cdot\mathbf{\Omega}'$ to just above the real $\mathbf{k}'\cdot\mathbf{\Omega}'$ axis along which the integral $\mathrm{d}^3\mathbf{J}'$ carries the integrand. As $\mathbf{k}'\cdot\mathbf{\Omega}$ passes just below $\mathbf{k}\cdot\mathbf{\Omega}$ the integral acquires a contribution $\mathrm{i}\pi$ times the numerator from the pole – see Figure 6.2 below and its discussion for more detail. Using this value of the integral, equation (6.48b) becomes

afterPlemelj
$$\begin{aligned}\mathbf{D}^{(2)}(\mathbf{J}) &= \pi(2\pi)^3 m\sum_{\mathbf{k}}\mathbf{k}\otimes\mathbf{k}\\ &\times\int\mathrm{d}^3\mathbf{J}'\sum_{\mathbf{k}'}\mathcal{P}|E_{\mathbf{k}\mathbf{k}'}(\mathbf{J},\mathbf{J}',\mathbf{k}'\cdot\mathbf{\Omega}')|^2 f_0(\mathbf{J}')\delta(\mathbf{k}\cdot\mathbf{\Omega}-\mathbf{k}'\cdot\mathbf{\Omega}').\end{aligned}\tag{6.49}$$

Our expressions for $\mathbf{F}^{(1)}$ and $\mathbf{F}^{(2)}$ now coincide with eqs (47) and (48) in [41].

tickleSec
6.4 Tickling systems

Tickling systems

In the last section we studied the normal modes of systems, linear combinations of which can model the dynamics of the system when it is left alone. Now we study the response of the system to being stimulated.

Any stimulus $\Phi_{\mathrm{e}}(t)$ can be Fourier analysed into a sum of sinusoidal stimulants, so it suffices to investigate what happens when we tickle the system at frequency ω. We will, however, need to allow ω to be complex. Moreover, we want to start tickling the system at a definite time, $t=0$, so we will work with Laplace rather than Fourier transforms.

Replacing $f_1(\mathbf{w},t)$ and $\Phi_1(\mathbf{x},t)$ in equation (6.1) by their Fourier expansions yields

$$\frac{\partial f_{\mathbf{k}}}{\partial t}+\mathrm{i}\mathbf{k}\cdot\mathbf{\Omega}f_{\mathbf{k}}=\mathrm{i}\mathbf{k}\cdot\frac{\partial f_0}{\partial\mathbf{J}}\Phi_{\mathbf{k}}.\tag{6.50}$$

We multiply by $\mathrm{e}^{\mathrm{i}\omega t}$ and integrate WRT t over $(0,\infty)$ to obtain

LTofCBEaa
$$\left[f_{\mathbf{k}}\mathrm{e}^{\mathrm{i}\omega t}\right]_0^\infty+\mathrm{i}(\mathbf{k}\cdot\mathbf{\Omega}-\omega)\widehat{f}_{\mathbf{k}}=\mathrm{i}\mathbf{k}\cdot\frac{\partial f_0}{\partial\mathbf{J}}\widehat{\Phi}_{\mathbf{k}},\tag{6.51a}$$

where the **Laplace transforms**[2] are

Laplace transforms!
$$\widehat{f}_{\mathbf{k}}(\mathbf{J},\omega)\equiv\int_0^\infty\mathrm{d}t\,\mathrm{e}^{\mathrm{i}\omega t}f_{\mathbf{k}}(\mathbf{J},t)\quad;\quad\widehat{\Phi}_{\mathbf{k}}(\mathbf{J},\omega)\equiv\int_0^\infty\mathrm{d}t\,\mathrm{e}^{\mathrm{i}\omega t}\Phi_{\mathbf{k}}(\mathbf{J},t).\tag{6.51b}$$

[2] A conventional Laplace transform is $\overline{y}(p)=\int_0^\infty\mathrm{d}t\,\mathrm{e}^{-pt}y(t)$. Our Laplace transforms coincide with conventional ones when one sets $p=-\mathrm{i}\omega$.