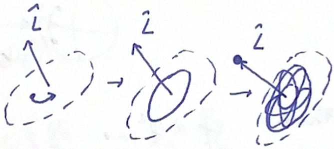
VRR and closure theory 15/07/2024

- BH + stars 

- very ≠ timescales → double orbit avg:



Hamiltonian: $\mathcal{H} = \sum_{i<j} U(\hat{L}_i \cdot \hat{L}_j)$

- only dyn. qty $\hat{L}(t)$, $\vec{L} = L\,\hat{L}(t)$
  L conserved: $K = (m, a, e) = \text{const.}$
  study of VRR ⟺ study of long term ev. of $\hat{L}(t)$.

-  N particles on the unit sphere.

- syst: stat. in time, isotropic, long-range interacting (large scale struct), correlated.

- stat. description: (discrete) DF $\quad f_d(\hat{L}, K, t) = \sum_{i=1}^{N} \delta(\hat{L} - \hat{L}_i(t))\,\delta(K - K_i)$

- Klimontovich eq: $\quad \frac{\partial f_d}{\partial t} + [f_d, \mathcal{H}] = 0$

- Q.L. expansion:
$$\begin{cases} f_d = \langle f \rangle + \delta f \\ \mathcal{H} = \langle \mathcal{H} \rangle + \delta \mathcal{H} \end{cases} \rightarrow \langle \ldots \rangle \text{ ensemble avg}$$

VRR: $\begin{cases} \langle f \rangle = \text{const} \\ \langle \mathcal{H} \rangle = 0 \end{cases} \rightarrow \boxed{\frac{\partial \delta f}{\partial t} + [\delta f, \delta \mathcal{H}] = 0} \; (*)$

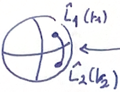
(*) → purely quadratic dyn., B.L. fails, no apparent timescale
dyn timescale
no frequencies, no phase-mixing
↳ $[\delta f, \langle \mathcal{H} \rangle] = 0$

- $1 = ((\ell, m), K, t)$
  position sphere, (m, a, e), time

Eq. of motion VRR: $f = \delta f \rightarrow \boxed{\frac{\partial f_1}{\partial t_1} = \frac{1}{2} \gamma_{123} f_2 f_3}$ "implicit sum"

non-random coupling coeff
↳ coupling btw ≠ scales, ≠ pops
captures sph. sym.

- Goal : compute correlation in time $\quad C_{12} = \langle f_1 f_2 \rangle$



$\leftarrow 1/\ell \quad ; \quad \ell \sim k \quad$ plasma

Casimir C_2 invariant
$\geq$ same time

- From $\quad \frac{df}{dt} \propto f^2 \rightarrow \frac{df}{dt} f \propto f^3 \xrightarrow[\langle \cdots \rangle]{} \frac{dC}{dt} \propto \underset{\substack{\uparrow \\ \text{closure pb.}}}{\langle fff \rangle}$

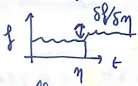
- General closure scheme : $\forall f$ DF, random field

$$\frac{df_1}{dt_1} = \underbrace{\gamma_1}_{\substack{\text{ext.}\\ \text{source}}} + \underbrace{\gamma_{12} f_2}_{\text{lin.}} + \frac{1}{2} \underbrace{\gamma_{123} f_2 f_3}_{\text{n.l.}}$$

VRR : $\gamma_1 = \gamma_{12} = 0 \quad \rightarrow \quad \gamma_{123} \equiv$ (vertex with legs 1, 2, 3) "bare" vertex

plasma: $\quad \partial_t f + \underbrace{\vec{v} \cdot \vec{\nabla} f}_{\text{lin.}} + \underbrace{\frac{q}{m} \left[\vec{E} + \frac{\vec{v} \times \vec{B}}{c}\right] \cdot \frac{\partial f}{\partial \vec{v}}}_{\text{n. l.} \quad (\vec{E}, \vec{B} \propto f)} = 0$

- Key idea, full description $\rightarrow$ consider syst's correlations and its response to inf. fluctuations.
  - $\rightarrow$ add perturb. eq of motion: $\quad \frac{df_1}{dt_1} = \cdots + \overset{\text{non-random}}{\eta_1}$
  - $\rightarrow$ define mean-response function : $\quad R_{12} = \left\langle \frac{\delta f_1}{\delta \eta_2} \right\rangle \Big|_{\eta = 0} \quad , \; t_1 > t_2 \;$ (causality)



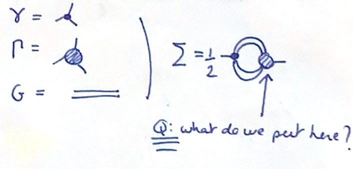
  - $\rightarrow$ look for self-consistent eqs for C, R. $\quad \rightarrow$ ~~$W[\eta] = \ln \langle \exp f\eta \rangle$~~

- MSR formalism (73) : based on QFT techniques, use of generating funct.

$$G_{12} = \begin{pmatrix} C_{12} & R_{12} \\ R_{21} & 0 \end{pmatrix} \xrightarrow[\text{closure}]{\text{MSR}} \boxed{\partial_{t_1} G_{12} = \underbrace{\delta_{12}}_{\text{source}} + \underbrace{\Sigma_{13} G_{32}}_{\text{propagator}}}$$

with : $\quad \begin{cases} \Sigma = \frac{1}{2} \gamma G G \Gamma \\ \Gamma = \gamma + \frac{\delta \Sigma}{\delta G} G G \Gamma \end{cases}$

$\uparrow$
renormalized vertex



- representation of R that gives the same structure as C:
$$C_{12} = \langle f_1 f_2 \rangle$$
$$R_{12} = \langle \_ \; \_ \rangle$$

- introduce new ~~operator~~ field $\hat{f}$ → $[f, \hat{f}] = \delta$ "conjugate field"

(skip tech. details of def of $\hat{f}$) ↑ analogous $[q,p] = i\hbar$

→ $R_{12} = \langle f_1 \hat{f}_2 \rangle$

→ Partition function: → generating funct. for correlations

$$Z[\eta, \hat{\eta}] = \int Df\, D\hat{f}\; \exp(-S + f\eta + \hat{f}\hat{\eta})$$

$$C_{12} = \left.\frac{\delta^2 Z}{\delta\eta_1 \delta\eta_2}\right|_{\eta=0} \qquad R_{12} = \left.\frac{\delta^2 Z}{\delta\eta_1 \delta\hat{\eta}_2}\right|_{\eta=0}$$

$$S = \int \mathcal{L}\,, \quad \mathcal{L} = \hat{f}\left(\partial_t f - \tfrac{1}{2}\gamma f f\right) \qquad \mathcal{H}(q,p)$$

why $\mathcal{L}$? $\qquad \mathcal{H}(f, \hat{f})$

$$\mathcal{H} = \hat{f}\dot{f} - \mathcal{L} = \tfrac{1}{2}\gamma \hat{f} f f \quad \rightarrow \partial_t f = \frac{\partial \mathcal{H}}{\partial \hat{f}} = \tfrac{1}{2}\gamma f f$$

→ $G_{12} = (C_{12}, R_{12})$ one qty that combines C, R

↳ $\partial_t C = \_\_$, $\partial_t R = \_\_$ $\Sigma(C,R)$ same structure

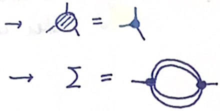
$$\partial_{t_1} G_{12} = \delta_{12} + \Sigma_{13} G_{32}$$

→ ψ fermion field, $\hat{\psi}$ anti-fermion field

→ if Z cannot be solved exactly → perturbative approach
↳ sum over Feyn. diags

- target for statistical closure: determine Γ (depends on 3-point functions)
- lowest-order closure → DIA (Kraichnan 58)

$$\begin{cases} \Gamma_{DIA} = \gamma \\ \Sigma_{DIA} = \frac{1}{2}\gamma G G \gamma \end{cases}$$



→ put $\Gamma = \gamma$ in MSR formula → eqs. $\begin{cases} \partial_t C = \dots \\ \partial_t R = \dots \end{cases}$ closed.

--- General closure scheme.

→ isotropic VRR → thermodynamic eq. (no ext. forcing, no dissipation)
tailor DIA to

↳ strictly conservative syst.

→ FDT holds → $C \propto R$.

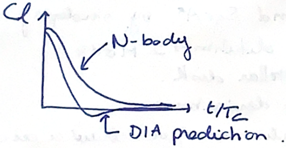
→ stat. + iso + single-pop: $C_{12} \propto C_{\ell_1}(t_1 - t_2)$

$$\partial_{t_1} C_{\ell_1} = \sum_{\ell_2, \ell_3} Q_{\ell_1 \ell_2 \ell_3} \int_0^{t_1} dt\, C_{\ell_2}(t_1 - t)\, C_{\ell_3}(t_1 - t)\, C_{\ell_1}(t) .$$

→ multi-scale, non-linear, isotropic

→ ~~scale-free in N → define a new rescaled time~~ (later)

↳



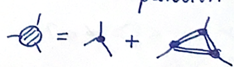
→ why doesn't it work? → DIA assumes quasi-Gaussian statistics

↳ not appropriate for VRR.

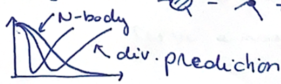
($f = f_G + \Delta f$, Δf small, $\cancel{\langle \Delta f^2 \rangle} 0$)

- Need to learn how to go to higher-order in the expansion

→ Naive expansion: $\Gamma[\gamma]$ →



↳ Divergence



↳ ill-behaved bc γ not a perturbative parameter.

↳ "let's do it and see" approach.

- Better possible expansion: $\gamma[\Gamma]$

→ more difficult to solve
→ find G and Γ simultaneously, fix point
→ to better understand.

Conclusion:

- VRR → archetype fully n.l / quadratic dyns at thermal eq.
- closure → compute cumulants any eq. of motion

$$\partial_t f_1 = \gamma_1 + \gamma_{12} f_2 + \gamma_{123} f_2 f_3$$

fluid / plasma / gravity
↳ (very ≠ regimes)

↳ Knowing C → characterizing dyn ∀ scale ℓ.

- DIA → lowest-order
  → need to go to next order, new expansion $\gamma[\Gamma]$
- Ideal Goal: systematic / explicit higher-order closure.

why recover N-body? → have a better feeling of the ≠ regimes
→ VRR: constrain n(K) stellar content around Sgr A* by studying pair dilution → IMBH's
↳ stellar disk.

- Lots of formalisms on closure, heuristic derivations
  ⟺ VRR test of those formalisms → |we draw a curve. |we measure something.
  ↳ isot + stat asymp.
  in this "degenerate" regime / "strong turbulence"

Big Picture

- what we don't understand:
  → expansion parameter, $\gamma[\Gamma]$, QFT clear α
  → how can we know in advance that closure will work?
  → what if $\langle f \rangle \neq 0$?
  → scale invariance w/ N?

$d_t f = -\partial_{\hat{L}} \cdot [f \, d\hat{L}/dt]$ / $\frac{d\hat{L}_i}{dt} = \vec{\Omega} \times \hat{L}_i$ / $T_c \propto \sqrt{N}$ / algo → second order accurate
$h = 10^{-2}$
error finite time $O(h^2)$

Side Notes Kocsis → time-correlated random walk $\quad H = \sum_{i<j} U(\hat{L}_i \cdot \hat{L}_j)$

- individual particles on the sphere have no individual kinetic energy.
  ↳ plasma diffusion in 2D in the presence of strong B

- VRR Hamiltonian: $H_{tot} = -\sum_{i<j} \sum_{\ell, m} H_\ell [K_i, K_j] \, Y_{\ell m}(\hat{L}_i) \, Y_{\ell m}(\hat{L}_j)$

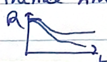
- reorientation driven by coherent grav torques btw stellar annuli.
- quasi Kep orbits (~ 10yr)
- pericenter prec: mean stellar pot + relativistic effects SMBH

- DIA realizable: realizability constraints associated with the existence of the PDF are preserved.
  → $\langle x^2 \rangle \geq 0$, $\langle x^2 \rangle \geq \langle x \rangle^2$, $\langle x^4 \rangle - \langle x^2 \rangle^2 \geq$ - - - -
  x rand. var.
  → Not obvious, for a moment-based closure → no guarantee that constraints are preserved in the course of time.

- BBGKY vs MSR → BBGKY expansion in N
  ↳ hierarchy that links one-particle f to $2, \ldots, N$-particle f.

- DIA realizable: $\exists$ PDF P s.t. $\langle \varphi_\alpha \varphi_\beta \rangle = \int \varphi_\alpha \varphi_\beta P$

- DIA: → Kraichnan: direct interactions of wave vectors $k + p + q = 0$ should dominate.
  → For strong turbulence DIA at best order-unity approx.
  → initial times OK → then prediction not good.



  → Kraichnan: DIA for homogeneous, isotropic turbulence at moderate Reynolds numbers. → satisfactory quantitative agreement

- DSR: Gaussian IC's + non-random γ $\rightarrow$ $\varphi_\alpha = \frac{1}{N}\sum_i Y_\alpha(\hat{L}_i)$ $\rightarrow$ $\langle \varphi_\alpha \varphi_\beta \varphi_\gamma \rangle_{t=0} \propto 1/N^2$
- ↳ amount of non-Gauss thrown away maps to order of truncation.
- path integral: partition function:

$$Z = \int D\varphi D\hat{\varphi} \langle \exp(-S + \varphi\eta + \hat{\varphi}\hat{\eta}) \rangle$$

where $[\varphi, \hat{\varphi}] = \delta$.

n-body functions
Bethe-Salpeter eqs
non-perturbative funct
renorm. group
↳ regulator action

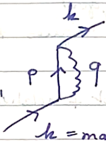
- Renormalization:

→ electron self-interaction:

$[q,p] = i\hbar$

$p = -i\hbar\, \partial_q$

factor $(p-m)^{-1}$



emission/absorption photon

(first order in α)

→ In general classical renormalization theory, the e^- propagator becomes the inf. response R, the photon term becomes C.

→ diag → divergent at large energies (integrate over the energies)

→ infinite e^- self-energy

↳ partial solution: mass renormalization:

"bare" mass m_0 should be renormalized by

(Srednik:) a correction δm due to the e^--radiation interaction:

scattering amplitude

$m = m_0 + \delta m$, $\delta m \propto \alpha$

↑ not exp. measured mass

Diag $= \int \frac{d^4k \ldots}{(p-m)\ldots}$

↳ regularizing the integrals of the interaction

↳ m instead of m_0 in the Lagrangian

↳ counter terms that cancel infinities

↳ analogous to charge renormalization

↳ solution to vacuum polarization by a charged particle.

→ | mass renormalization: e.g. of Propagator renormalization | G

| charge renormalization: e.g. of Vertex renormalization. | Γ

- weak turbulence → $\Delta f \ll f_G$ (?)
- f_G → Gaussian pdf. ($\neq$ from f_G being a Gaussian (Maxwellian))
- would closure work better in linear regime? → scaling γ_{12}/γ_{123} possible.