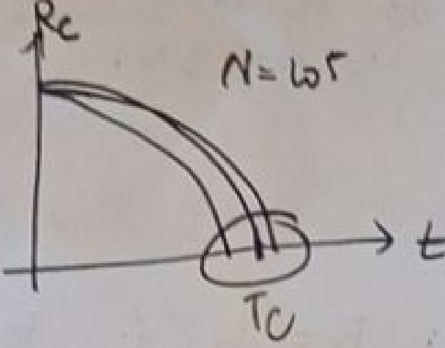
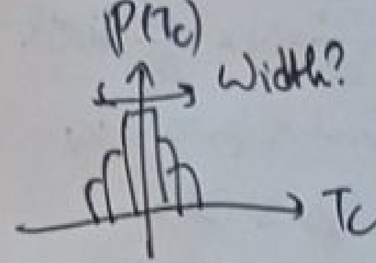
Dynamical large deviations for the Landau equation

with Ouassim Feliachi

Core Collapse of a GC :

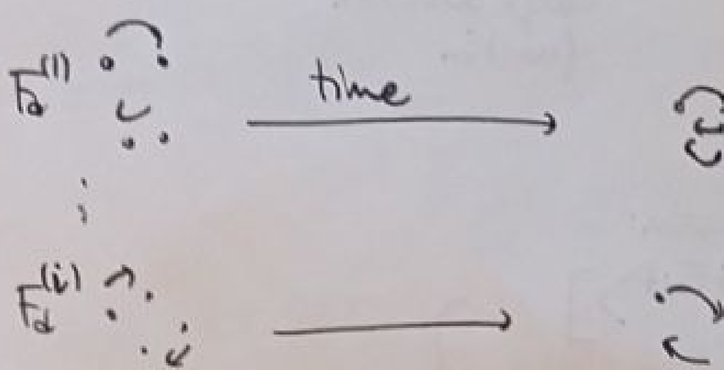




Ensemble-average

Phase-space

$F_d = \sum_i m\, \delta_D[w - w_i(t)]$ $\quad \frac{\partial F_d}{\partial t} + [F_d, H_d] = 0$

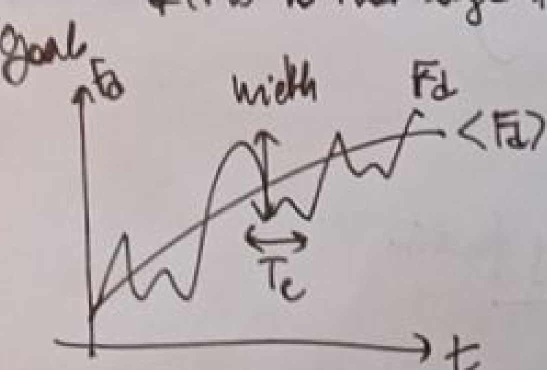


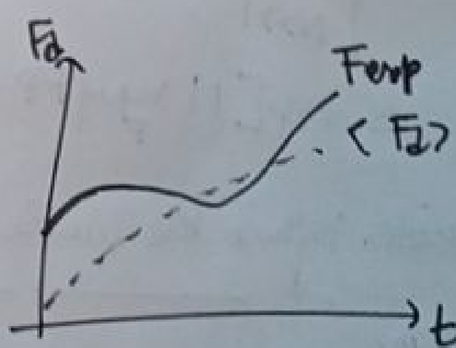
$\langle F_d \rangle$ $\xrightarrow{\text{Kinetic theory}}$: $\frac{\partial \langle F_d \rangle}{\partial t} = \frac{\partial \langle F_d \rangle}{\partial t}[\langle F_d \rangle]$ No stochasticity

What is missing?

+ N is not that large + Single Observation + What about unlikely evolutions?

Goal





$\mathbb{P}[F_d = F_{exp}]$?

Faking dynamics: $\frac{\partial F_d}{\partial t} = \frac{\partial \langle F_d \rangle}{\partial t}[F_d] + \underbrace{\eta[F_d]}_{\text{Noise}}$ $\quad \langle \eta \eta \rangle$: { Amplitude? Time correlation? Spatial Correlation?

QL expansion

$F_d = \langle F_d \rangle + \delta F$ & $\delta F \ll \langle F_d \rangle$ $\longrightarrow$ $\frac{\partial \langle F_d \rangle}{\partial t} + \langle [\delta F, \delta \phi] \rangle = 0$ $\boxed{\text{Non-stochastic!}}$

Ensemble averaged DF

Angle-averaged DF:

$F_N(J,t) = \int \frac{d\theta}{(2\pi)^d} F_d(\theta, J, t)$ $\longrightarrow$ $\begin{cases} \langle F_N \rangle = \langle F_d \rangle \\ F_N \xrightarrow{N \to \infty} \langle F_d \rangle \end{cases}$

We write $F_d = F_N + \delta F$

$\boxed{\text{Stochastic!}}$

with $\delta F \ll F_N$, we get a slow-fast system: $\boxed{\tau = \frac{n}{N}}$: secular time

$\frac{\partial \delta F}{\partial \tau} = N \times \frac{\partial \delta F}{\partial \tau}[F_N, \delta F]$ & $\frac{\partial F_N}{\partial \tau} = \frac{\partial F_N}{\partial \tau}[F_N, \delta F]$

Evolves for $\tau \sim 1/N$; Evolves for $\tau \sim 1$

~~Spatial Chaos~~

Large deviation Principle:

- A scalar variable: $F_N = \frac{1}{N}\sum_1^N f_i$ For $N \to \infty$, $\langle F_N \rangle \to \langle f \rangle$
  What about the PDF?

LDP: $\mathbb{P}[F_N = F] \simeq \exp[-N \; I[F]]$

Large deviation rate (N); Large deviation function ($I[F]$)

~~give ourselves a conjugate value~~

(Scaled) Cumulant generating function:

for P given: $\mathcal{H}[P] = \frac{1}{N} \ln[\langle e^{N F_N P} \rangle]$

How to compute $\mathcal{H}[P]$:

$$\langle e^{N F_N P} \rangle = \int dF \, \mathbb{P}(F_N = F) \, e^{NFP}$$
$$= \int dF \, e^{-NI(F)} e^{NFP}$$
$$= \int dF \, e^{N(FP - I(F))}$$
$$\downarrow N \gg 1$$
$$\simeq \exp[N \sup_F \{FP - I(F)\}]$$

$\mathcal{H}[P] = \sup_F \{FP - I[F]\}$

$\downarrow$ Legendre transform

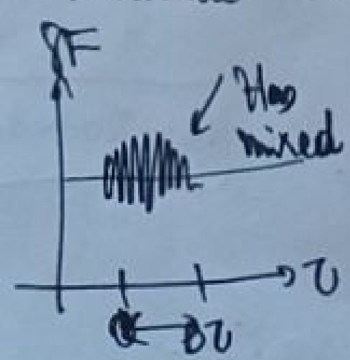
$\boxed{I(F) = \sup_P \{FP - \mathcal{H}(P)\}}$

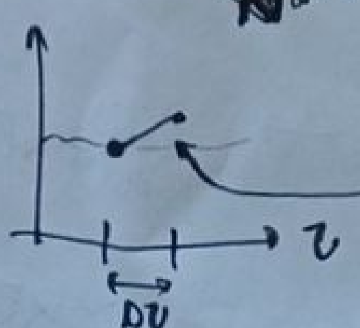
Result: Large deviation require the cumulant generating function.

Now with ~~...~~ Landau

$\frac{\partial \delta F}{\partial \tau} = N \frac{\partial \delta F}{\partial \tau}$; $\frac{\partial F_N}{\partial \tau} = \frac{\partial F_N}{\partial \tau}$ $\tau = \frac{t}{N}$ ($\sim \frac{1}{\sqrt{N}}$)

Introduce: $\Delta\tau$ such that: [redacted] $\frac{t_{dyn}}{N} \ll \Delta\tau \ll t_{dyn}$





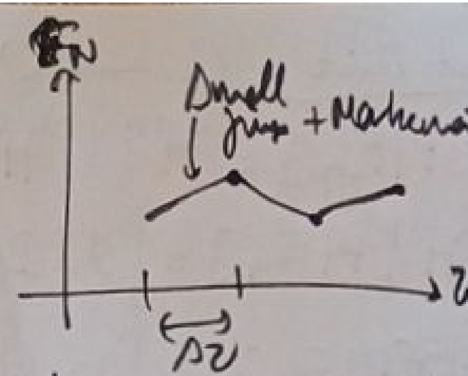
Consider the function

$$G_N = \frac{F_N(\tau + \Delta\tau) - F_N(\tau)}{\Delta\tau}$$

③

Scheme of the computation:

$$G_N = \frac{1}{N} \int_0^{\Delta\tau} dt\, \partial_\tau F_N \qquad \underline{\text{Slow . fast}}$$



$$\mathbb{P}\left(\{F_N(\tau)\}_{0\le\tau\le T} = \{F_{exp}(\tau)\}_{0\le\tau\le T}\right)$$

$$\simeq \prod_n \mathbb{P}\left(G_N(n\Delta\tau) = \dot{F}_{exp}(n\Delta\tau) \,\middle|\, F_N(n\Delta\tau) = F_{exp}(n\Delta\tau)\right)$$

<u>Now</u>: $\mathbb{P}\left(G_N = \dot{F}_{exp} \mid F = F_{exp}\right) = e^{-N\Delta\tau\, I(\dot{F}_{exp} \mid \dot{F}_{exp})}$

with $I(\dot{F}_{exp} \mid \dot{F}_{exp}) = \sup_P \left\{ \dot{F}_{exp} P - \mathcal{H}(P \mid F_{exp}) \right\}$

with $\mathcal{H}(P \mid F_{exp}) = \frac{1}{\Delta t} \ln\left[\langle e^{\Delta t\, G_N P} \rangle_{F_{exp}}\right]$

<u>Finally</u>: $G_N = \frac{1}{\Delta\tau} \int_0^{\tau} d\tau\, \partial_\tau F_N = \frac{1}{\Delta t} \int_0^{\Delta t} dt\, \partial_\tau F_N$

Ansi: $\mathcal{H}(P \mid F_{exp}) = \lim_{\Delta \to +\infty} \frac{1}{\Delta} \ln\left[\left\langle \exp\left[P \int_0^{\Delta} dt\, \partial_\tau F_N\right]\right\rangle\right]$

<u>Finally, generic results</u>:

$$\mathbb{P}\left[\{F_N = F_{exp}\}_{0\le\tau\le T}\right] \simeq \cancel{\exp\left\{-N \sup_P \left[\int_0^T d\tau \{\right.\right.}$$

$$= \exp\left[-N \sup_P \left[\int_0^T d\tau \left[\int dJ\, \dot{F}_{exp} P - \mathcal{H}[P \mid F_{exp}]\right]\right]\right]$$

large deviation rate

$P = P(J, \tau)$ conjugate field

$\dot{F}_{exp} = \partial_\tau F_{exp}$: prescribed

large deviation Hamiltonian

$$\mathcal{H}[P \mid F_{exp}] = \lim_{\Delta \to +\infty} \frac{1}{\Delta} \ln\left[\left\langle \exp\left(\int_0^{\Delta} dt \int dJ\, P(J)\, \underbrace{\partial_\tau F_N[\delta F]}\right)\right\rangle_{F_{exp}}\right]$$

$\partial_\tau F_N \simeq -[\delta F, \delta\phi(\delta F)]\rangle$

i.e. Gaussian random field

Gaussian integral!

In the hot limit: $G \to 0$

then $\mathcal{H}[P|F_{exp}] = \mathcal{H}^{(1)} + \mathcal{H}^{(2)} + o(G^2)$

with: $\mathcal{H}^{(1)}[P|F_{exp}] = \int dJ\, P(J) \left(\frac{\partial F_{exp}}{\partial z}\right)_{Landau}$

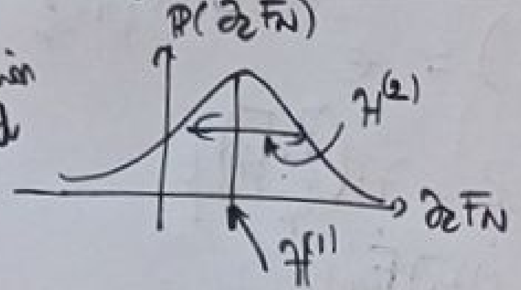
$$\mathcal{H}^{(2)}[P|F_{exp}] = \int dJ\, dJ'\, P(J)\, P(J')\, \underbrace{Q[F_{exp}](J,J')}_{\geq 0}$$

Use linear propagator + $\delta F(0)$ gaussian with $\langle \delta F(w)\, \delta F(w') \rangle = \frac{1}{N} \delta(w-w') f(w)$

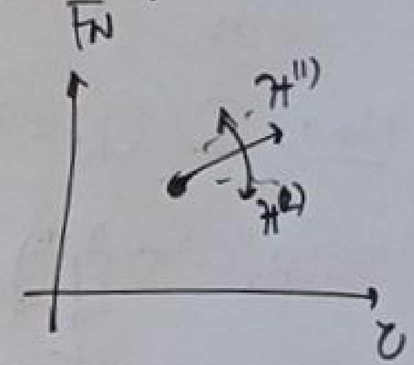
Only two cumulants to compute

~~Properties of the large deviation Hamiltonian.~~

→ gaussian field



Flow of the DF



Properties of the large deviation

+ Most probable path: $\frac{\partial F_N}{\partial z} = \frac{\delta \mathcal{H}}{\delta P}\Big|_{P=0} = \left(\frac{\partial F_N}{\partial z}\right)_{Landau}$

+ Conserves Mass & Energy: $\int dJ\, \frac{\delta C(F)}{\delta F(J)} \cdot \frac{\delta \mathcal{H}}{\delta P(J)} = 0$

+ (Hamilton-Jacobi, time symmetry, gradient structure)

Stochastic PDE:

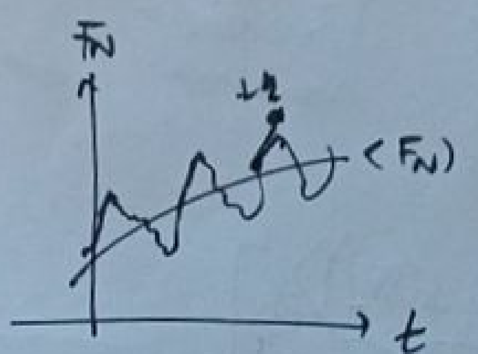
Fokker-Planck ⟷ Langevin

$\frac{\partial F(v)}{\partial t} = \frac{\partial}{\partial v}\left[-D_1 F + \frac{1}{2} \frac{\partial}{\partial v}(D_2 F)\right]$ ⟷ $F = \langle v_i \rangle$

with: $\Delta v_i = \underbrace{D_1 \Delta t}_{\text{Deterministic drift}} + \underbrace{\sqrt{D_2 \Delta t}\, \xi(t)}_{\text{Stochastic diffusion}}$ $\quad \langle \xi \rangle = 0$, $\langle \xi(t) \xi(t') \rangle = \delta(t-t')$

From deviation to Langevin:

$\frac{\partial F_N}{\partial z} = \left(\frac{\partial F_N}{\partial z}\right)_{Landau} + \eta[F_N](z)$ with: $\begin{cases} \langle \eta \rangle = 0 \\ \langle \eta(z)\, \eta(z') \rangle = \frac{1}{N} Q\, \delta(z-z') \end{cases}$



Diagonal rewriting

$\eta = \frac{1}{\sqrt{N}} Q^{1/2} \underbrace{\xi(z)}_{\text{Normal}}$

& exactly complies with conservation laws

Next steps:

- Collective effects ($G \gg 0$) (All cumulants needed)
- Marginal stability: $G = 1 \pm \epsilon$
- Derive from BBGKY
- Evaluation in practice (cube then Gaussian)
- Fake dynamics for real (Monte-Carlo)
- Kinetic blockings $\mathcal{H}=0$ in 1D