Vector Resonant Relaxation

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Down memory lane

SEGAL meeting — Edinburgh April, 2023

Vector Resonant Relaxation

Particular thanks to the following graduate students

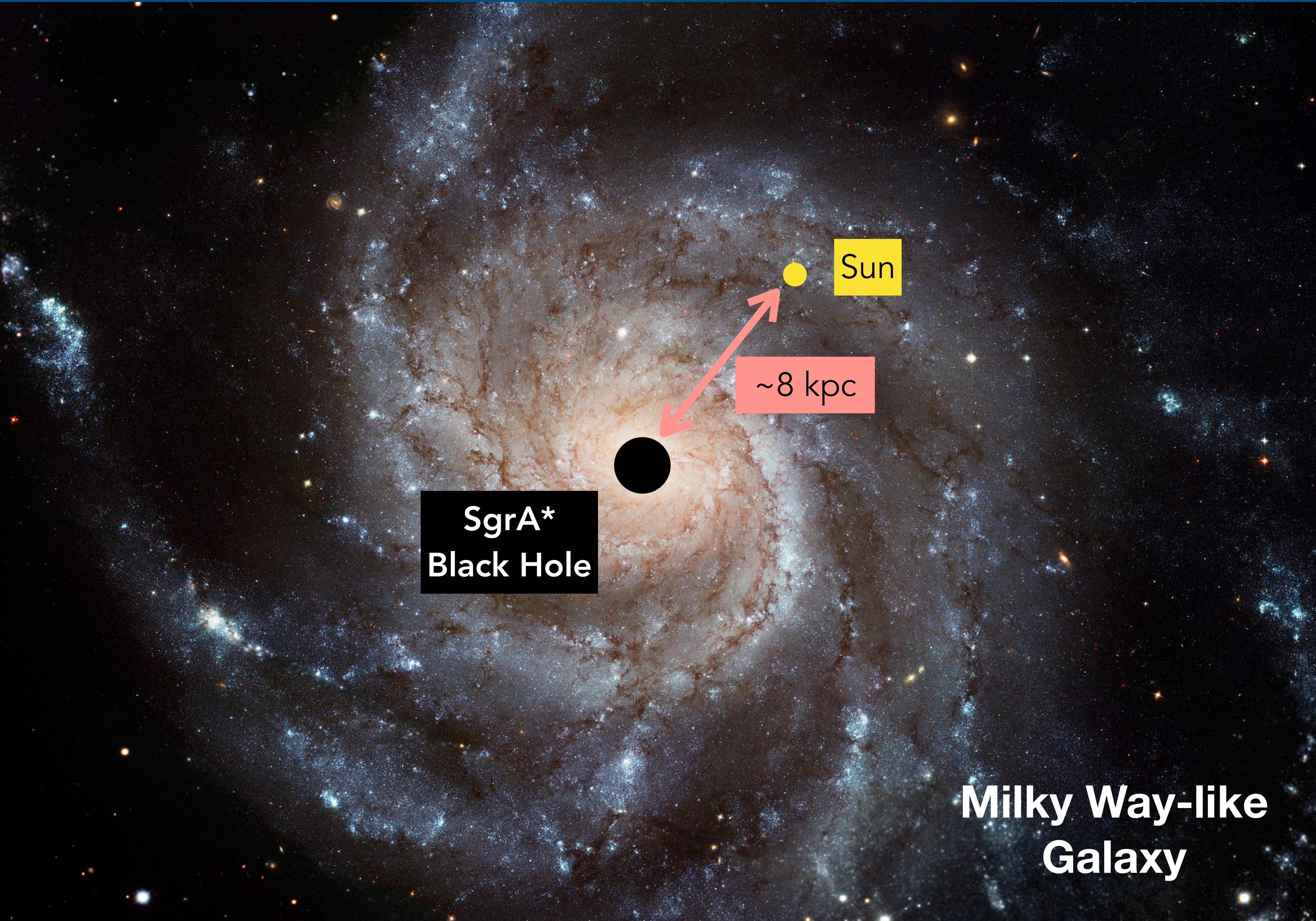
Sofia Flores
Renormalisation

Kerwann Tep
Eccentricity relaxation

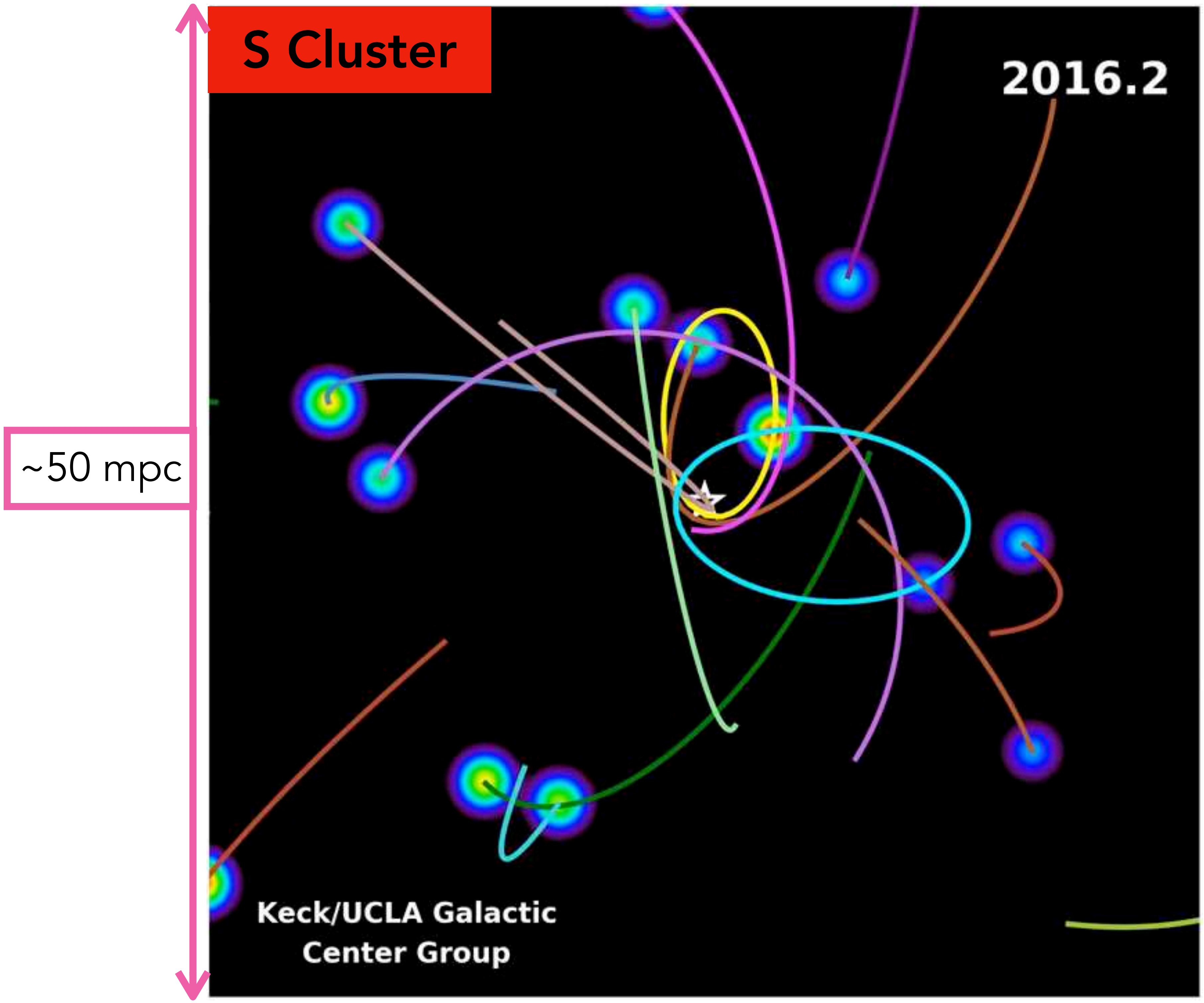
Mathieu Roule
Kinetic blockings

Juan Giral
Neighbor separation

Nathan Magnan
Thermodynamical Eq.


Sun
~8 kpc
SgrA*
Black Hole
Milky Way-like
Galaxy

SgrA*, at the heart of the Milky Way

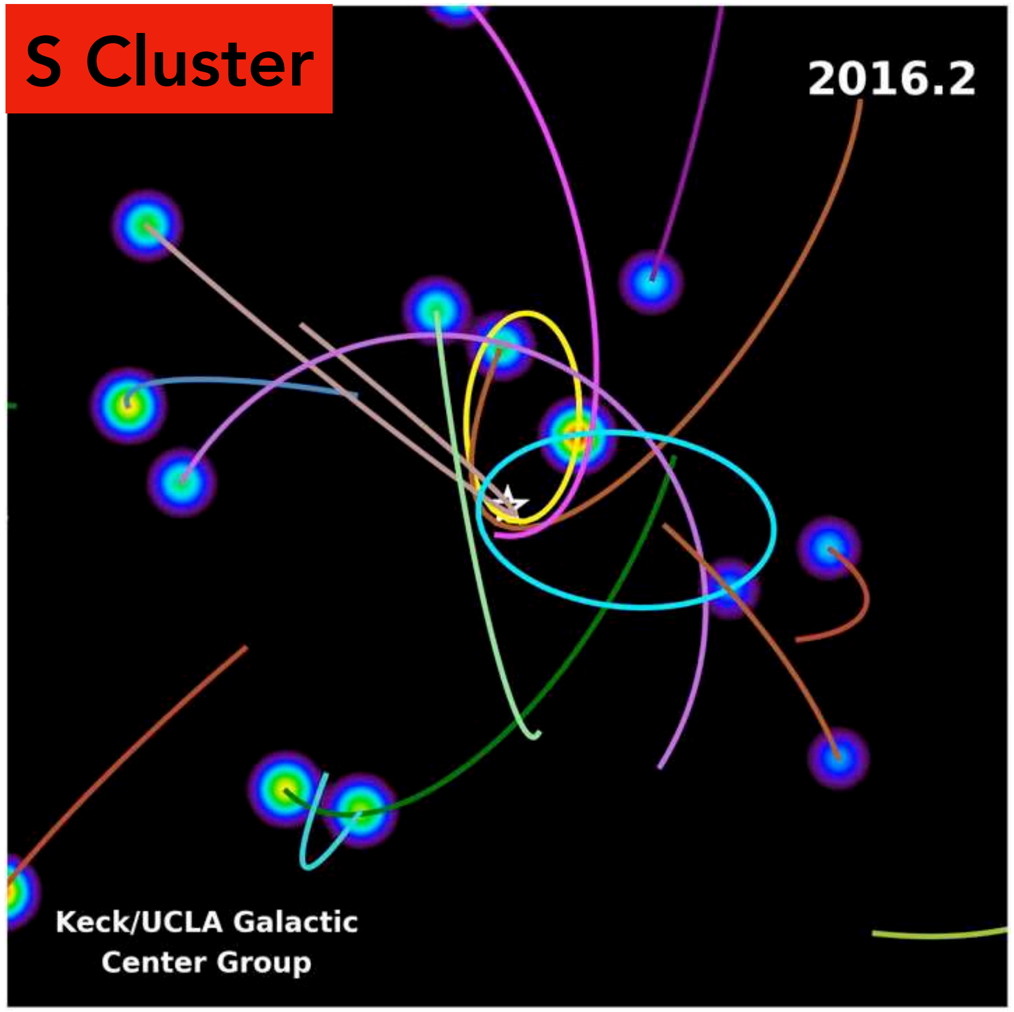


What is the dynamics around **supermassive black holes**?

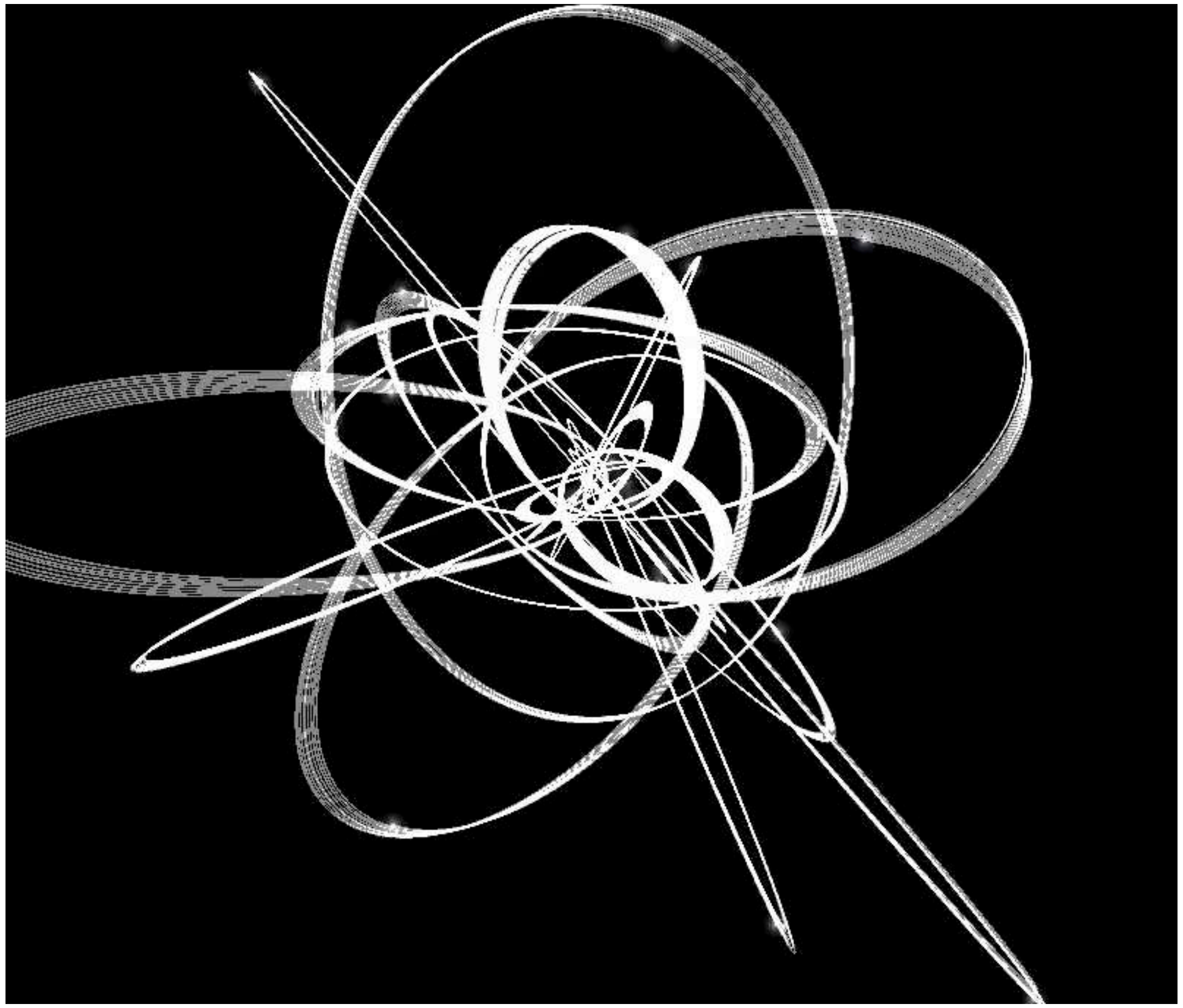
A simple dynamics?

The central BH is **supermassive**

$$M_{\mathrm{SgrA}} \simeq 4{,}200{,}000 \times M_{\mathrm{Sun}}$$ vs. $$M_{\mathrm{Sun}} \simeq 330{,}000 \times M_{\mathrm{Earth}}$$



Keck observations

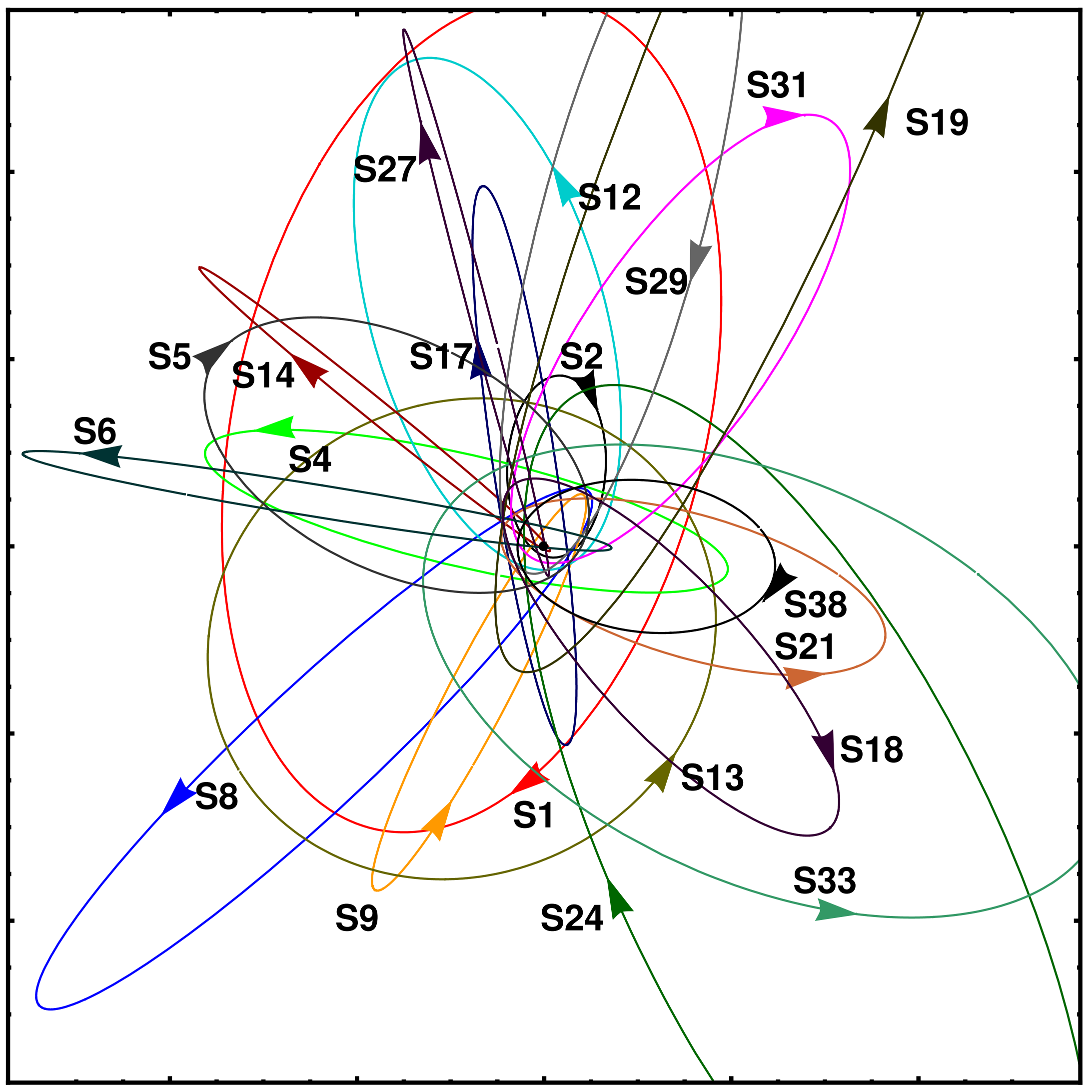


Numerical simulations

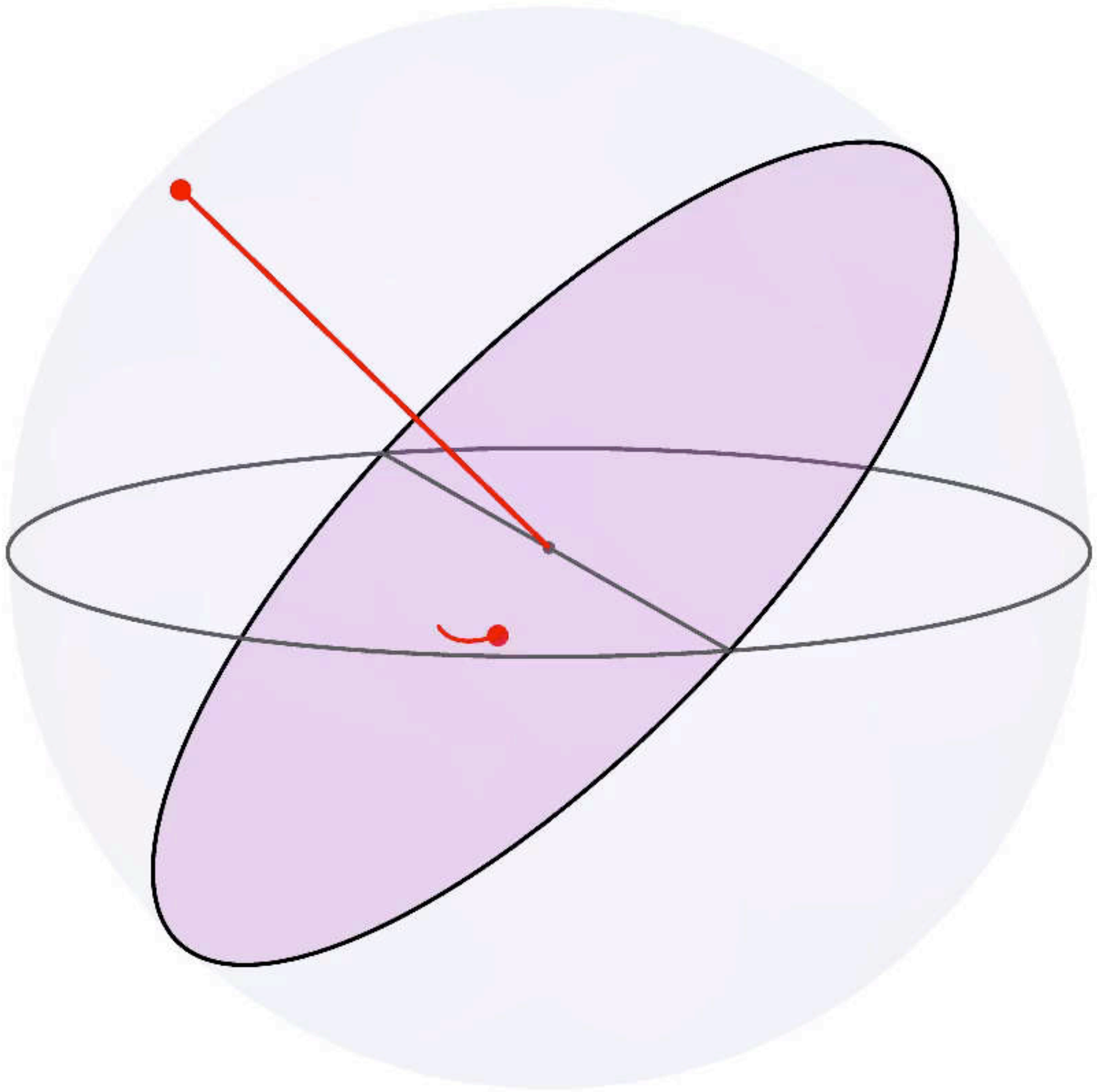
Like the Earth around the Sun, stars follow **Keplerian orbits**

Keplerian orbits

The BH dominates the stars' dynamics



Gillessen et al., 2009



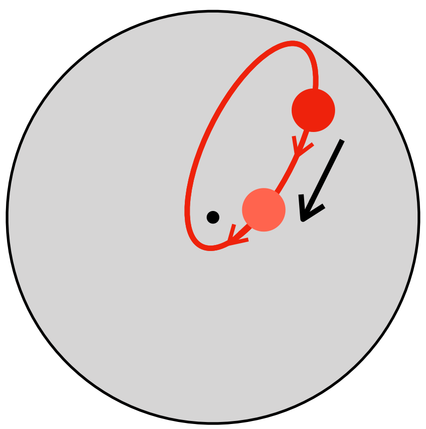
VLT observations

Typical orbit

What is an orbit?

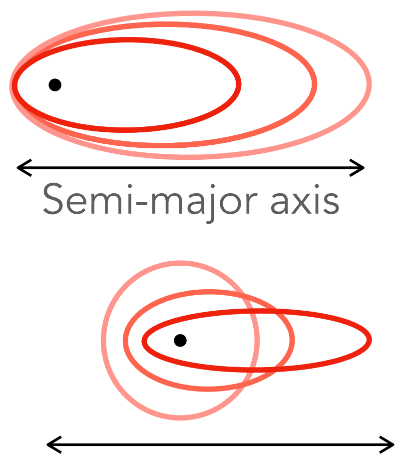
Describing an **orbit**

Position of the star

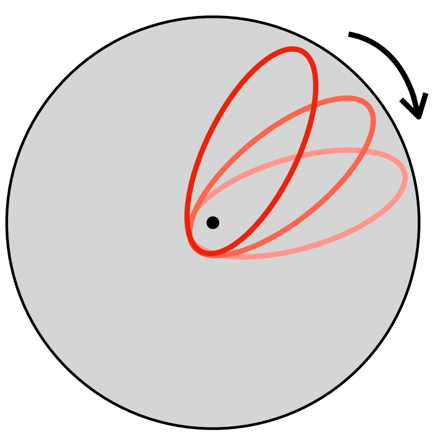


Dynamical motion

Shape of the orbit

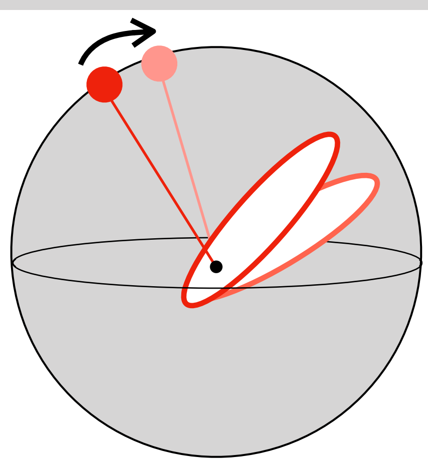


Phase of the orbit

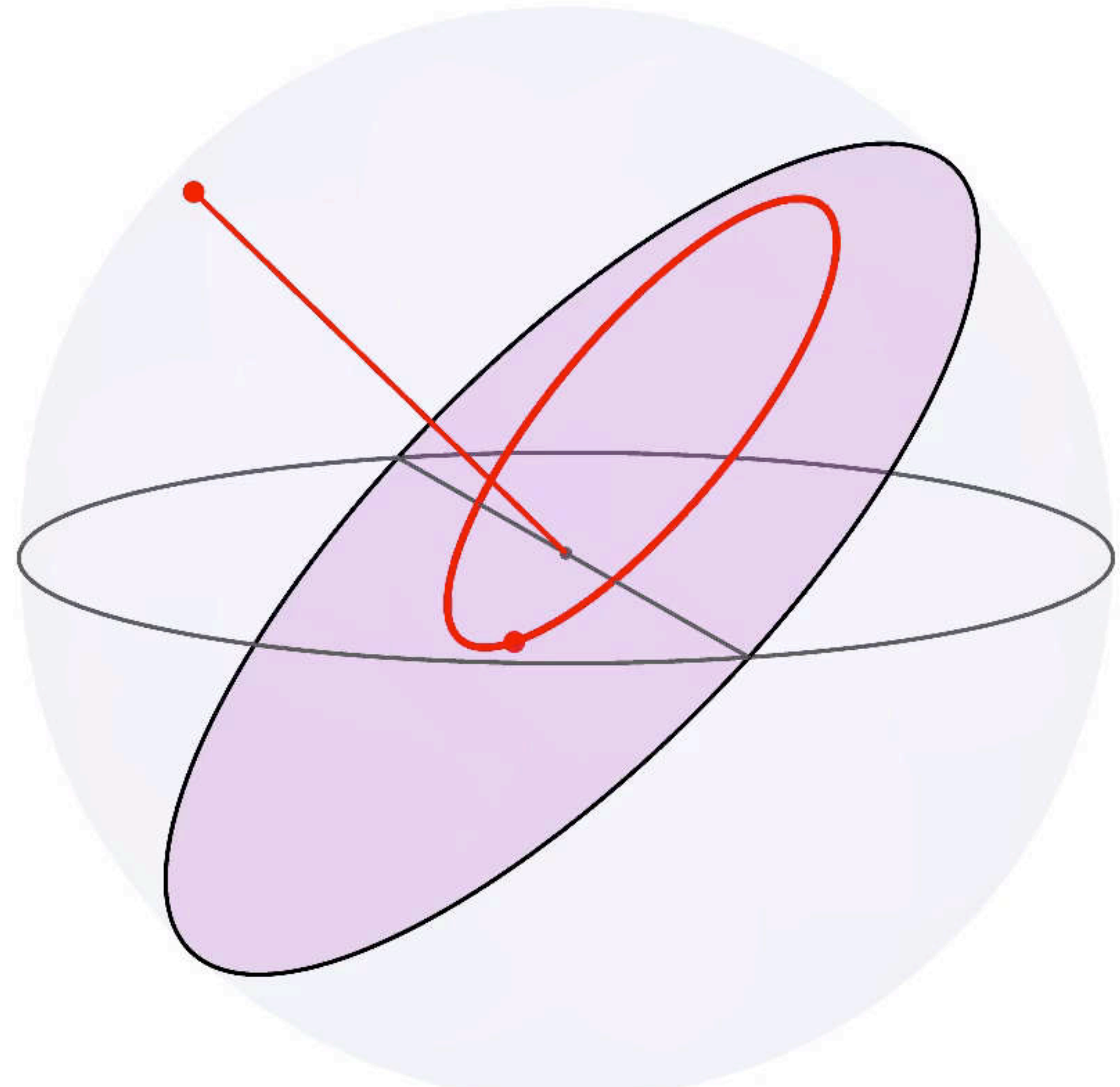


Phase of the pericentre

Orientation of the orbit



Spatial orientation



Keplerian orbit

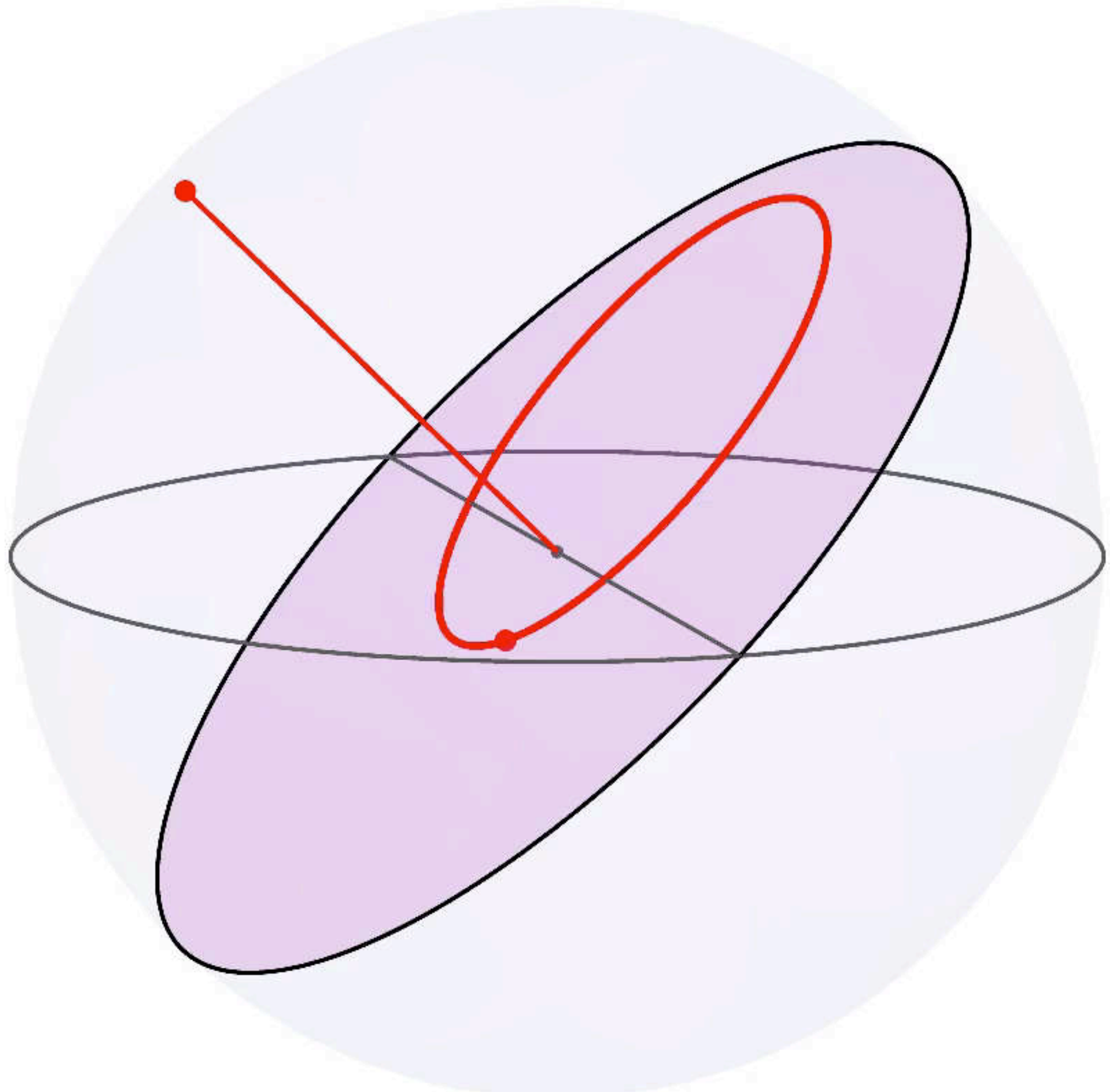
What is the dynamics of **Keplerian orbits**?

Pericentre precession

Origins of the **precession**:

+ **Relativistic** effects from the BH
+ **Perturbations** from other stars

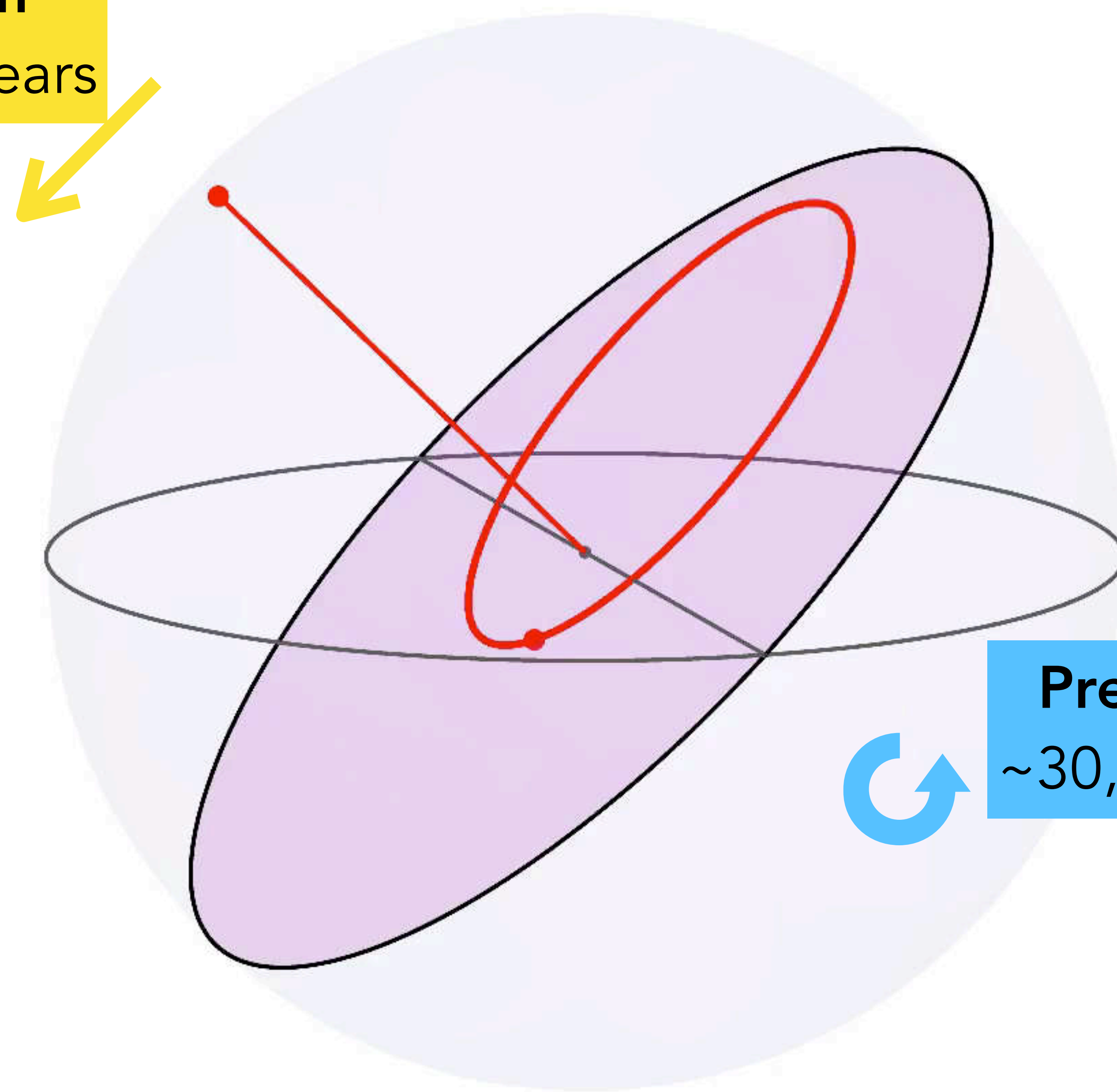
~30,000 years for S2



Orbits **precess** in their planes

Orbits also change in orientations

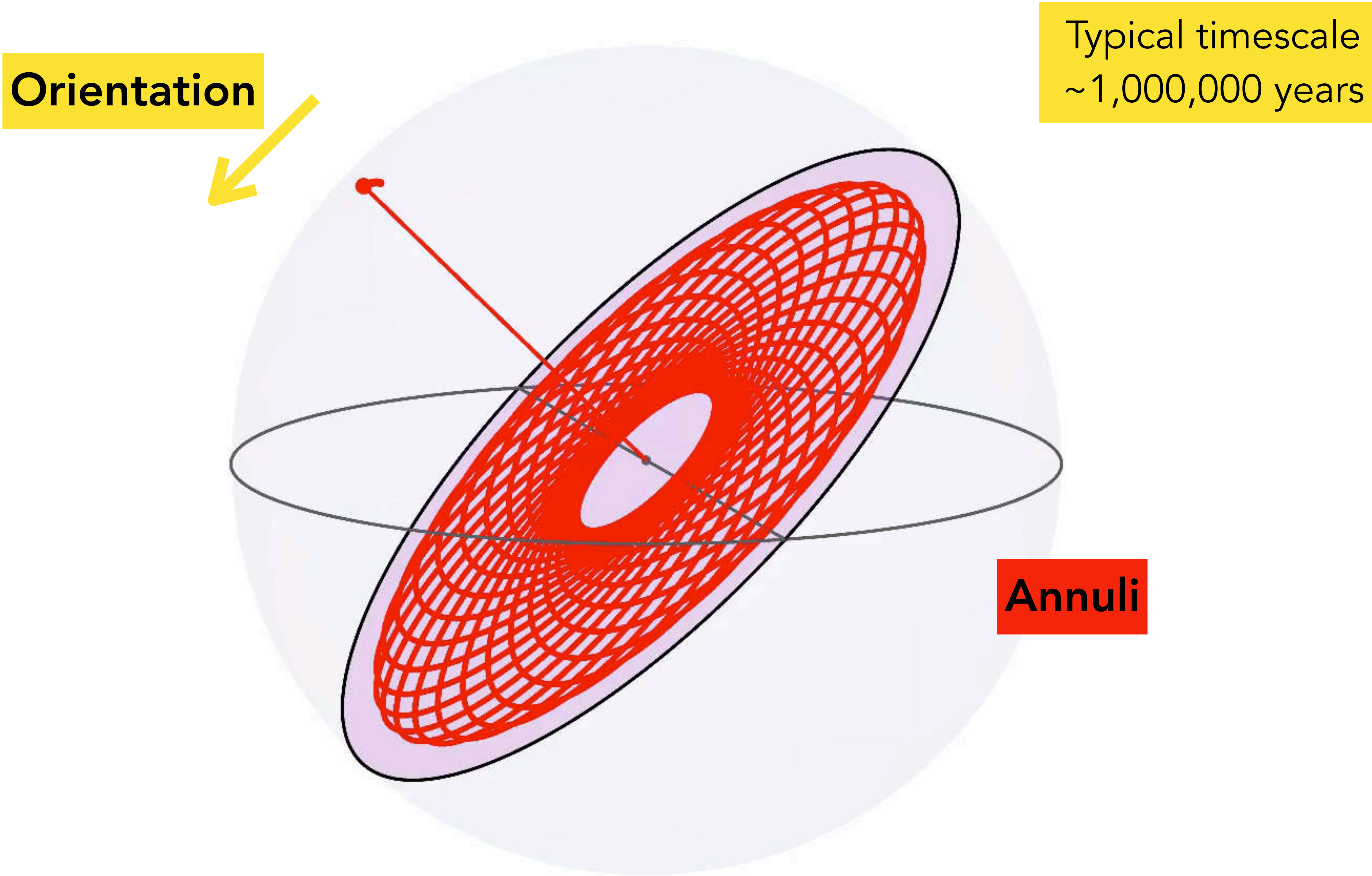
Orientation
~1,000,000 years



Precession
~30,000 years

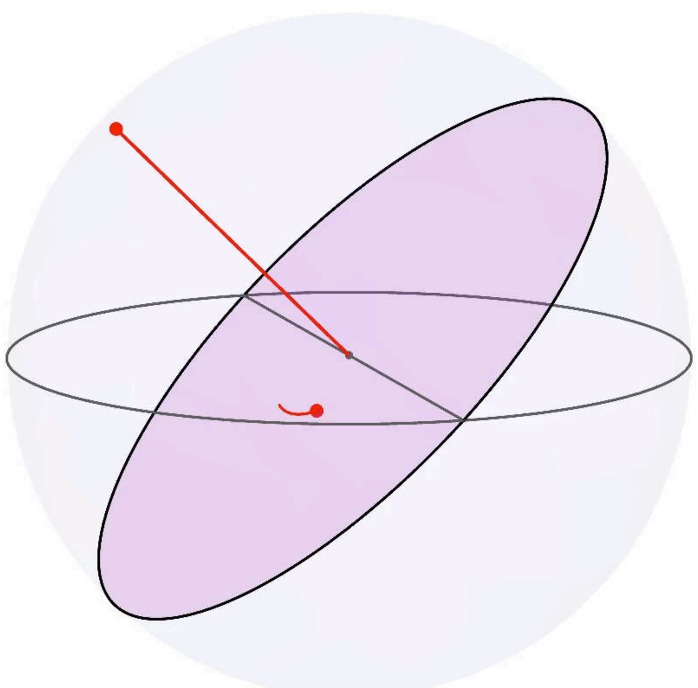
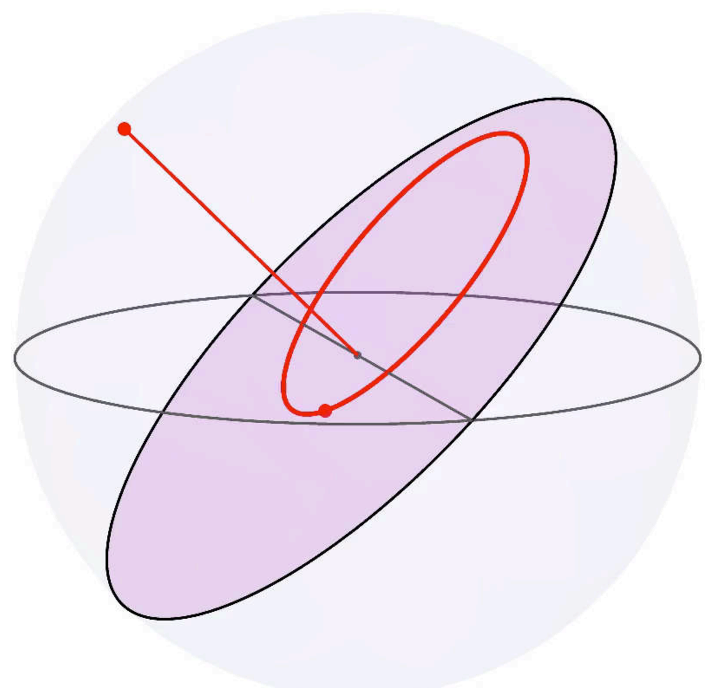
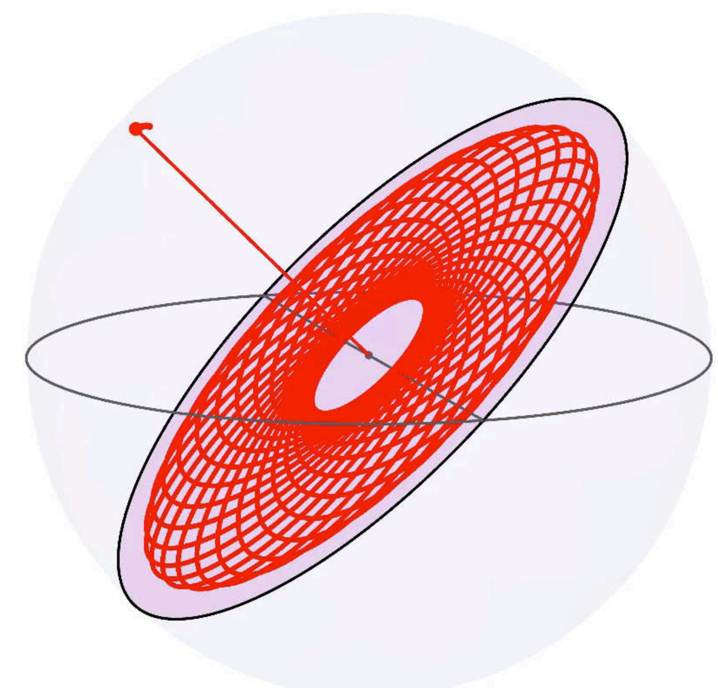
Two timescales: **Precession** ≪ **Orientation**

Stellar orientations



After a full precession, **ellipses** become **annuli**

Stellar dynamics

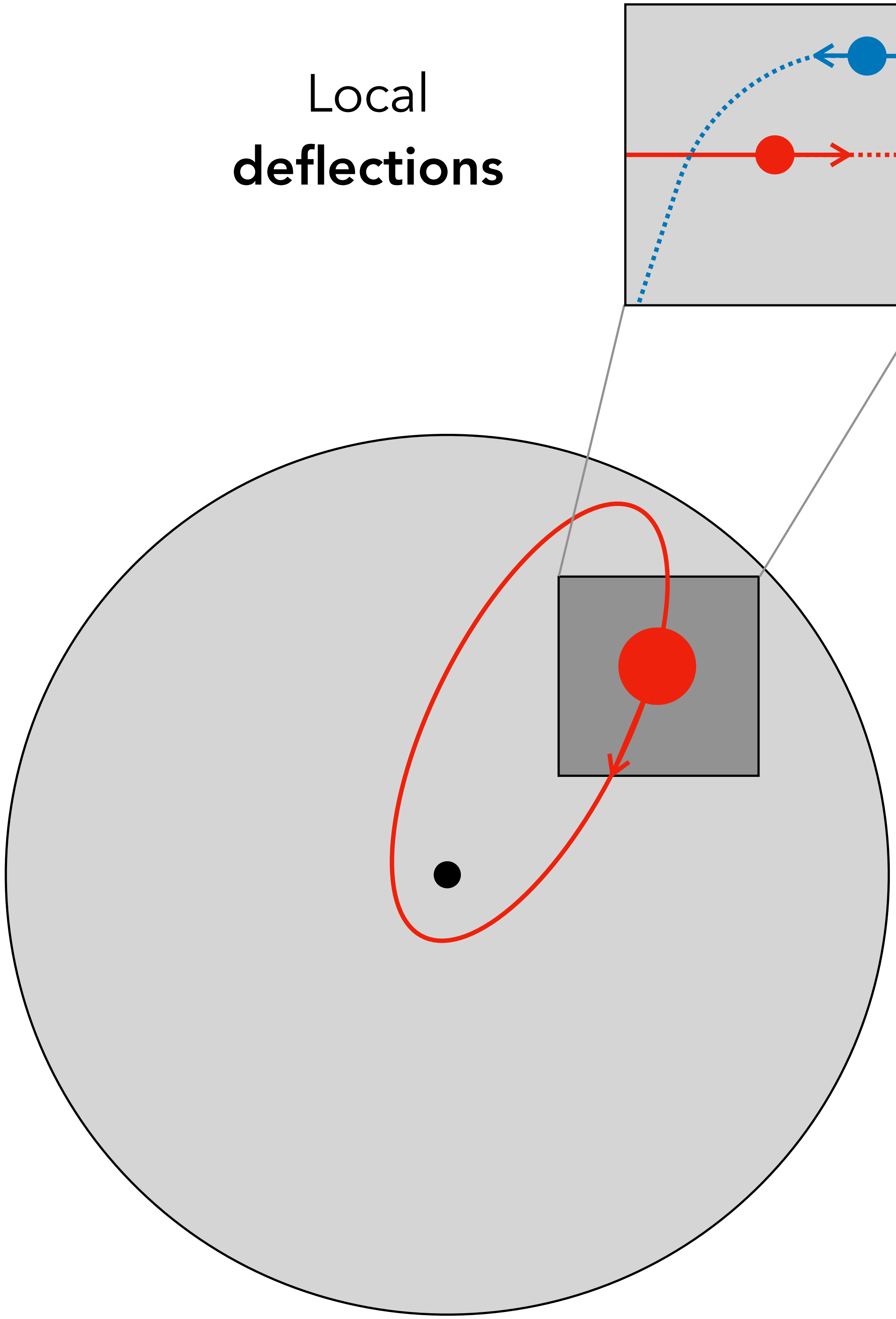
Stars	Ellipses	Annuli
		
~10 years	30,000 years	~1,000,000 years
Orbital motion	**Pericentre** precession	**Orientation** precession

SgrA* is 10 Gyr orld. We can wait longer

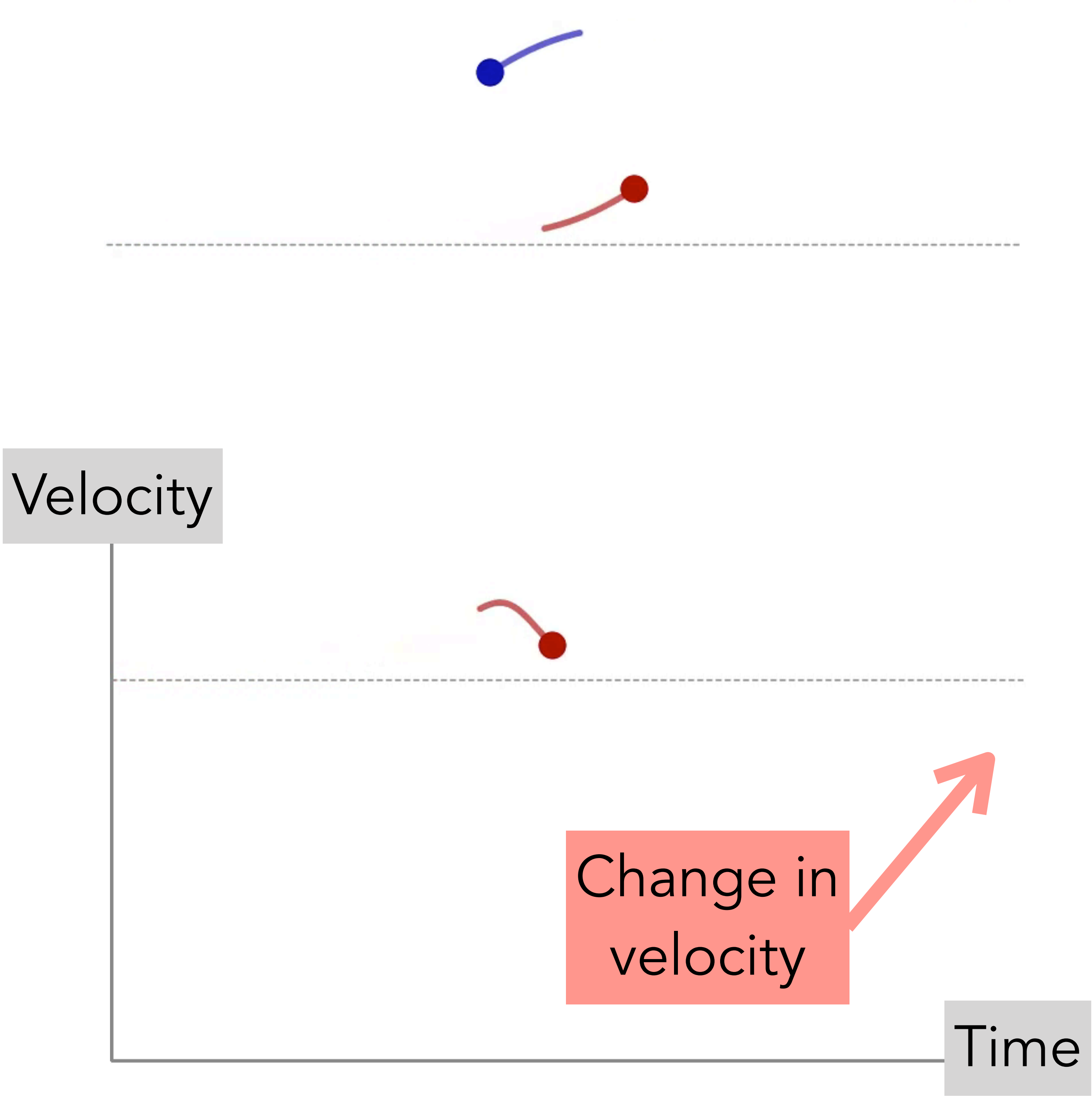
Stellar energy

Orbital distortions sourced by instantaneous **kicks and deflections**

Local
deflections

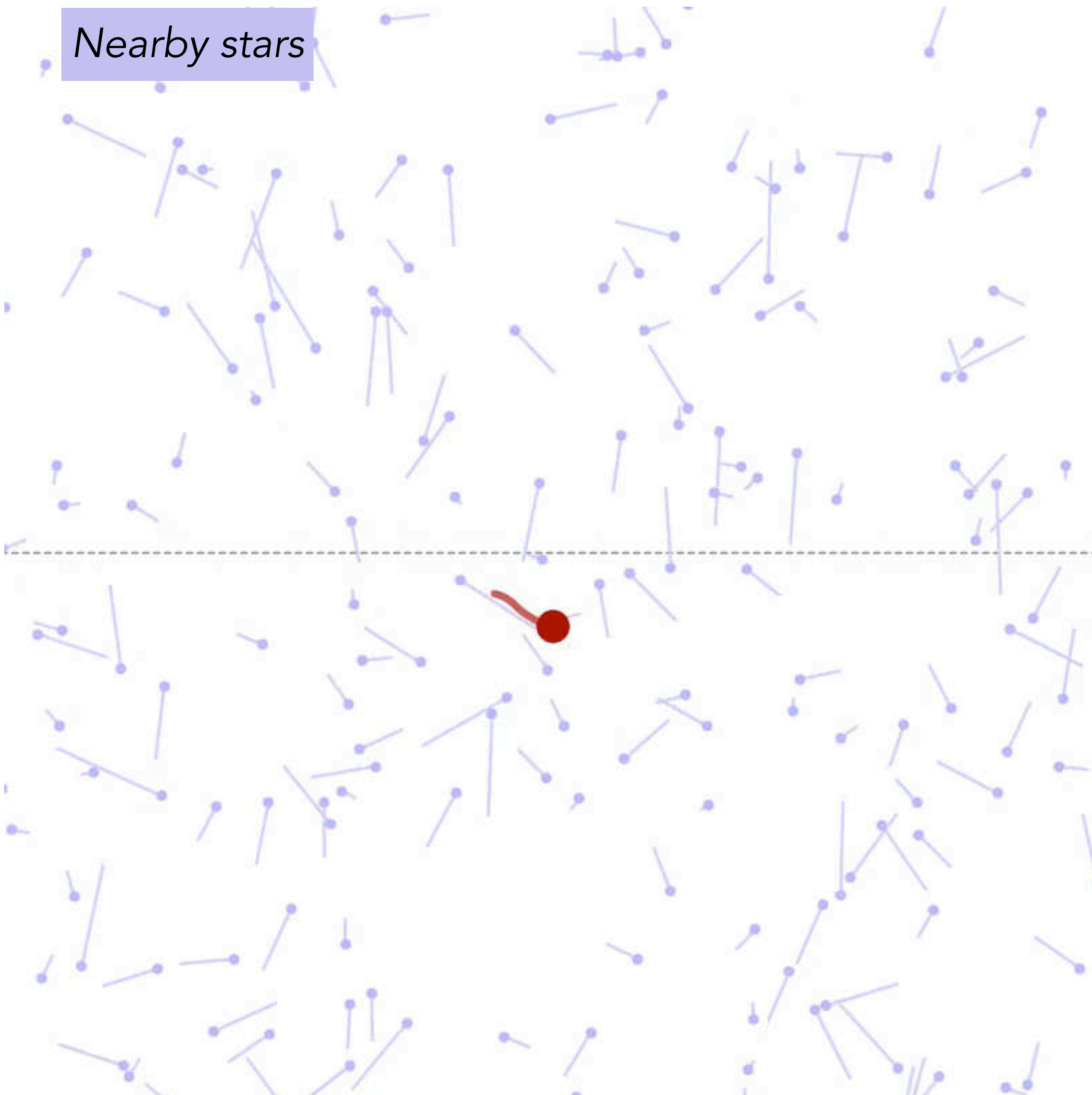


Zoom on the orbit

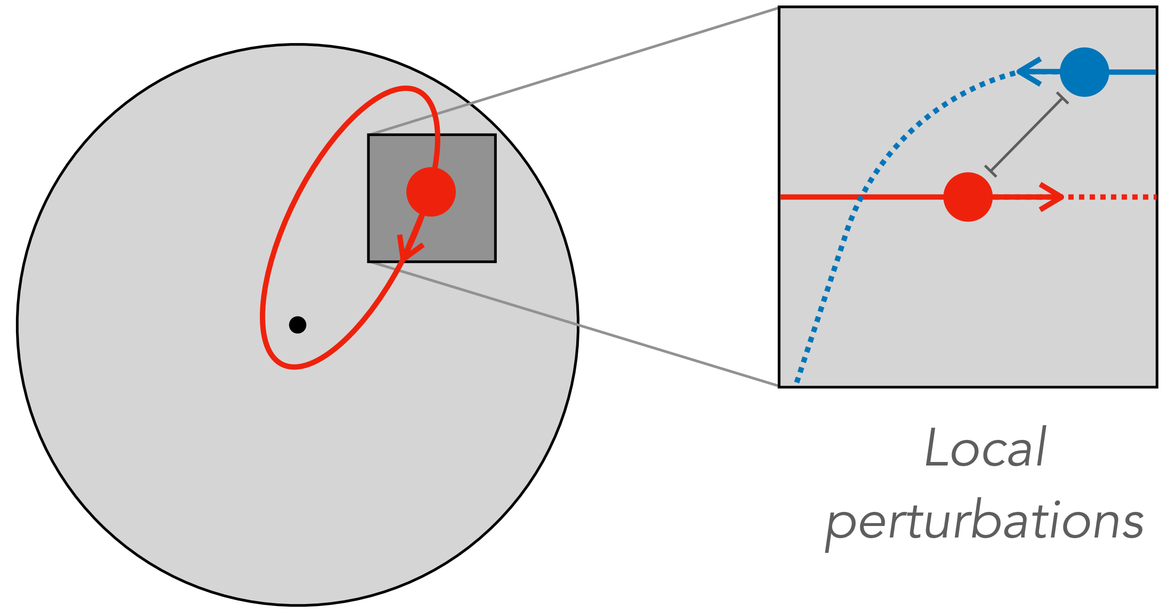


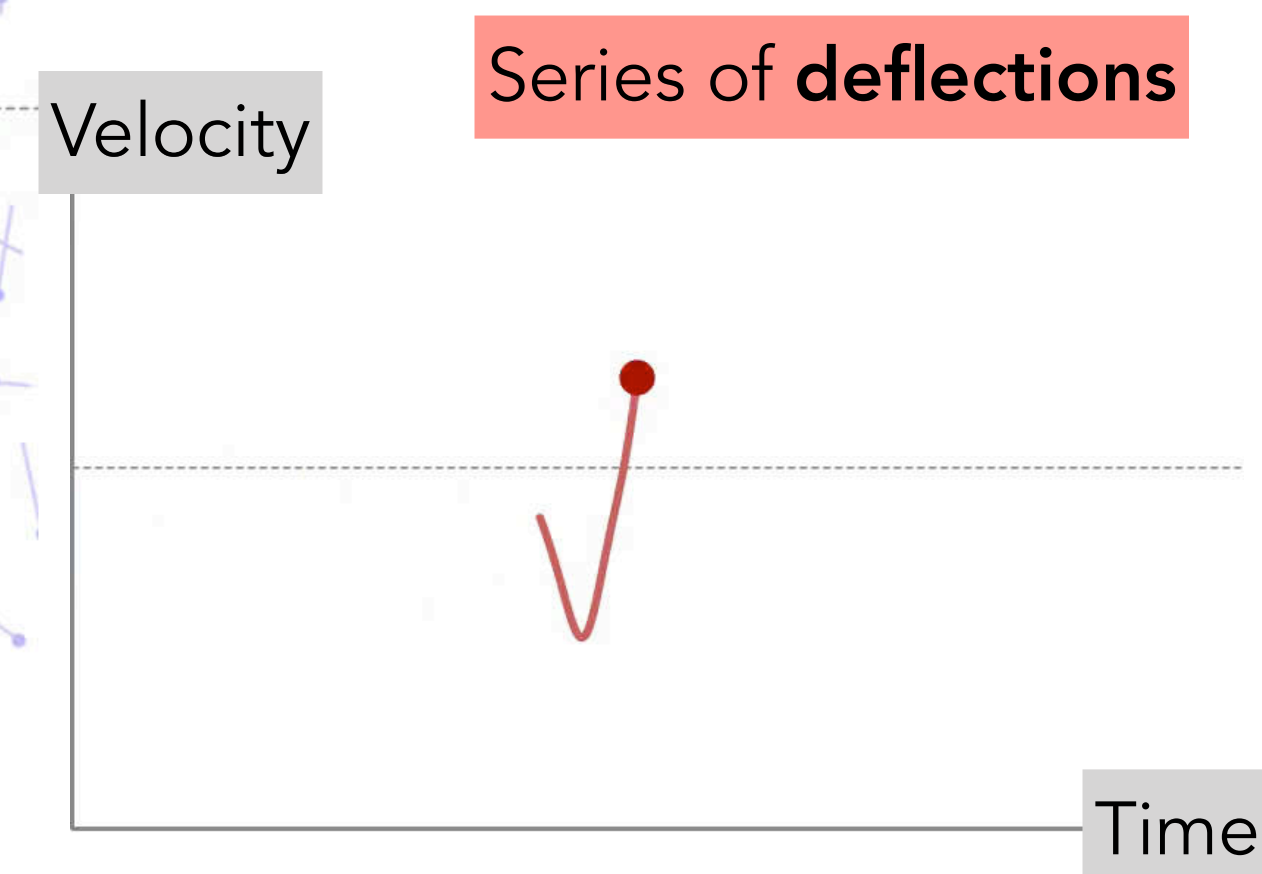
Deflections

The star has a lot of **close neighbours**



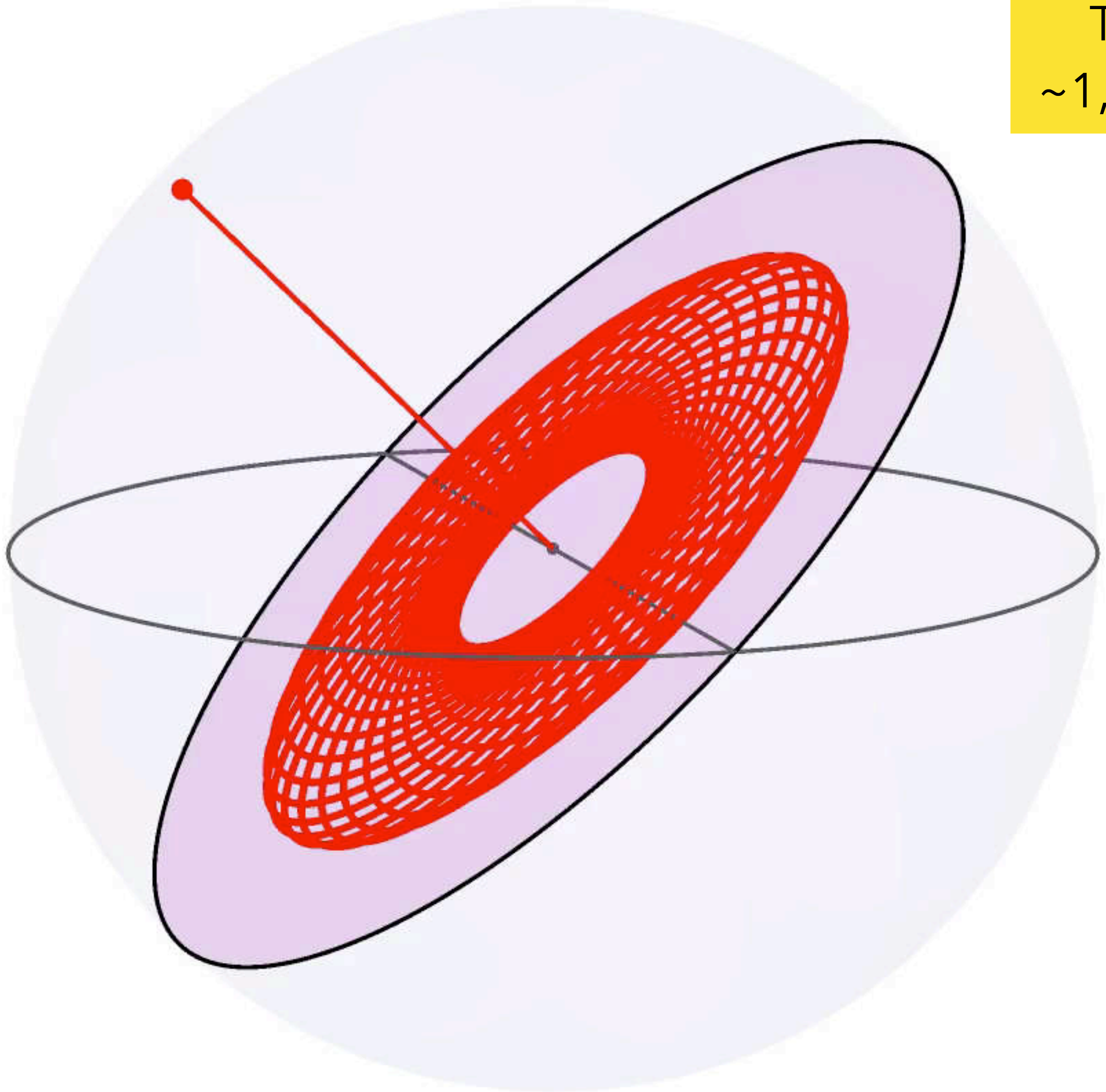
Random walk





Stellar energy

Typical timescale
~1,000,000,000 years



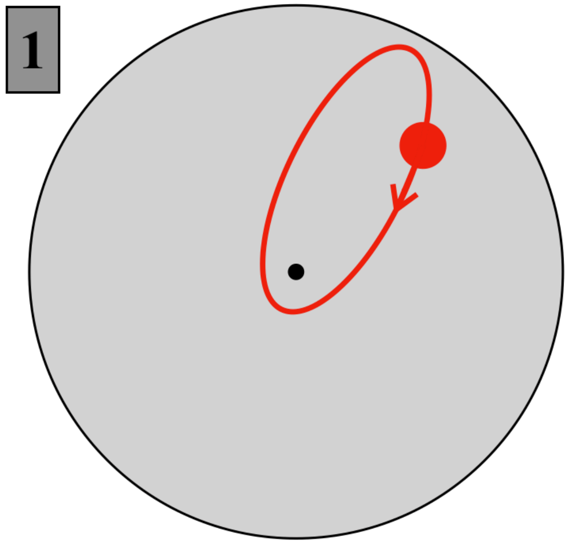
Deflections drive a slow change in the Keplerian energy

Timescales are highly hierarchical

1. Dynamical time

Fast orbital motion induced by the BH

$$\frac{\mathrm{d}M}{\mathrm{d}t} = \Omega_{\mathrm{Kep}}$$



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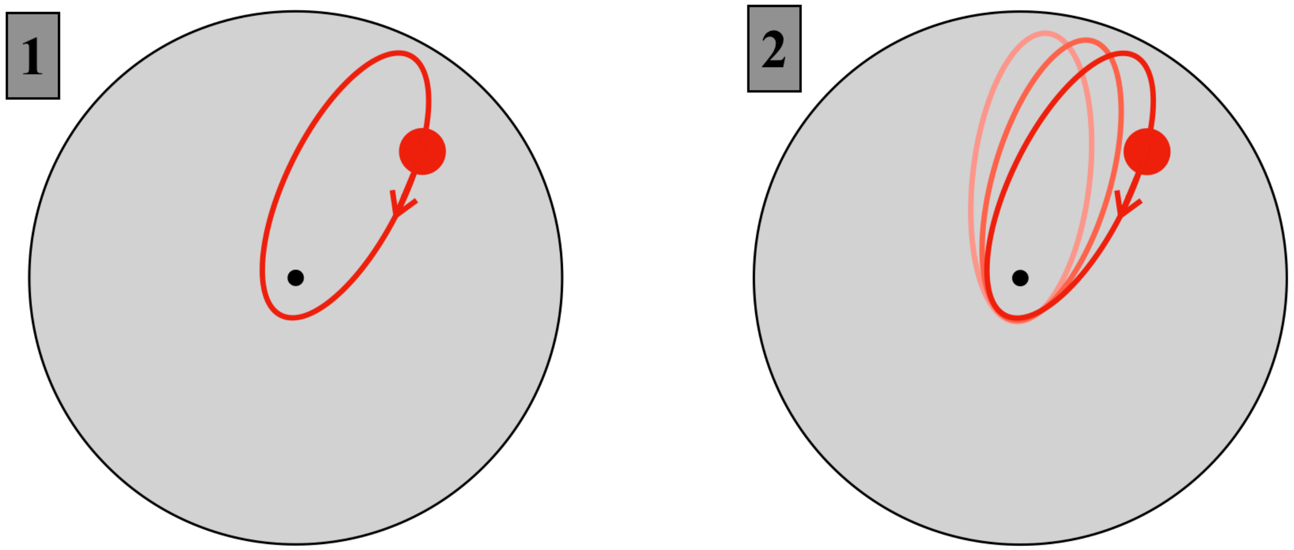
2. Precession time

In-plane precession (mass + relativity)

$$\frac{\mathrm{d}\omega}{\mathrm{d}t} = \Omega_{\mathrm{p}}$$



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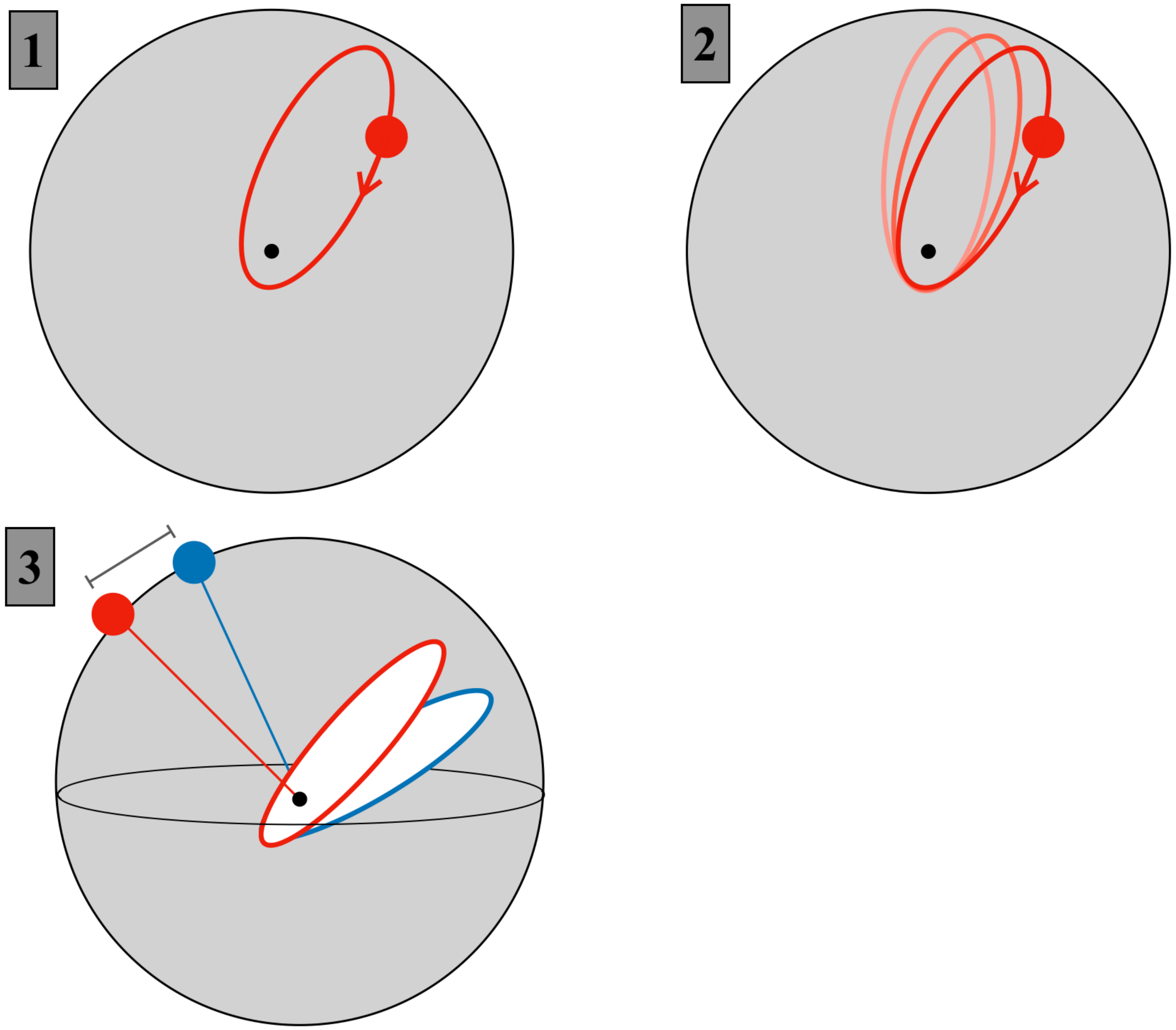
3. Vector Resonant Relaxation
Non-spherical torque coupling

$$\frac{\mathrm{d}\hat{\mathbf{L}}}{\mathrm{d}t} = \eta(\hat{\mathbf{L}}, t)$$



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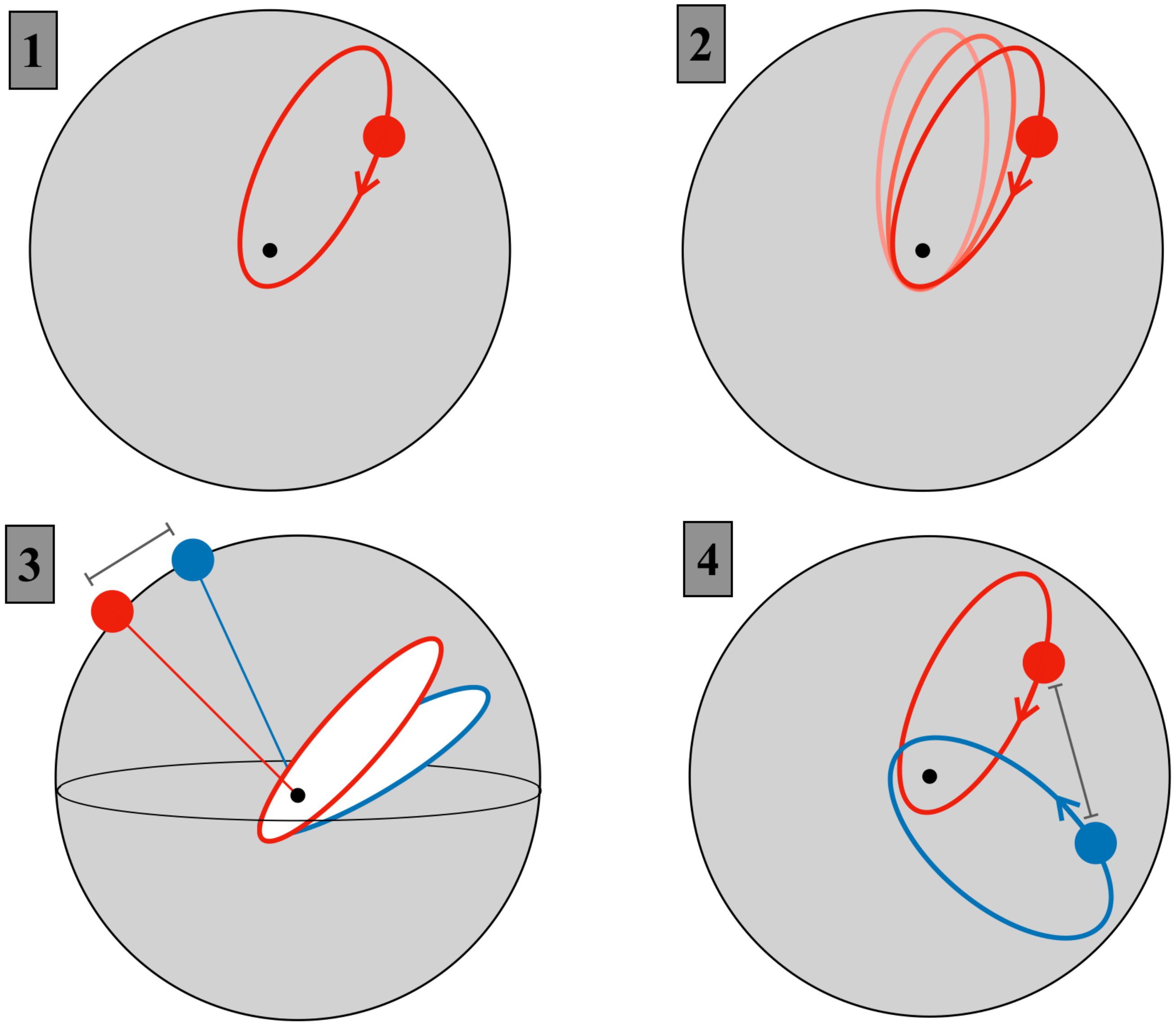
$$\frac{\mathrm{d}\hat{\mathbf{L}}}{\mathrm{d}t} = \eta(\hat{\mathbf{L}}, t)$$

4. Scalar Resonant Relaxation
Resonant coupling on precessions

$$\frac{\mathrm{d}e}{\mathrm{d}t} = \eta(e, t)$$



Timescales are highly hierarchical

1. Dynamical time
Fast orbital motion induced by the BH

$$\frac{\mathrm{d}M}{\mathrm{d}t} = \Omega_{\mathrm{Kep}}$$

2. Precession time
In-plane precession (mass + relativity)

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3. Vector Resonant Relaxation
Non-spherical torque coupling

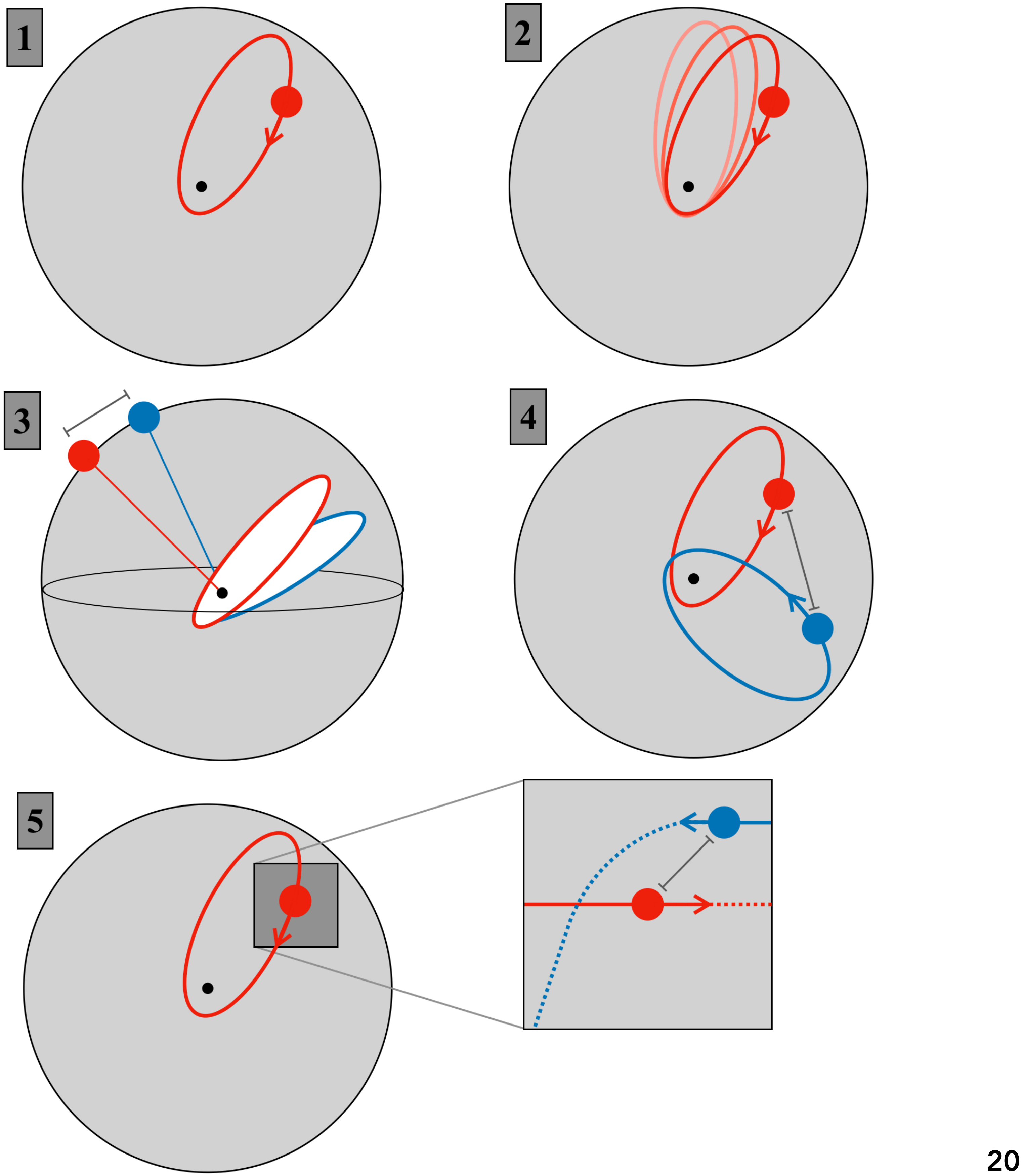
$$\frac{\mathrm{d}\widehat{\mathbf{L}}}{\mathrm{d}t} = \eta(\widehat{\mathbf{L}}, t)$$

4. Scalar Resonant Relaxation
Resonant coupling on precessions

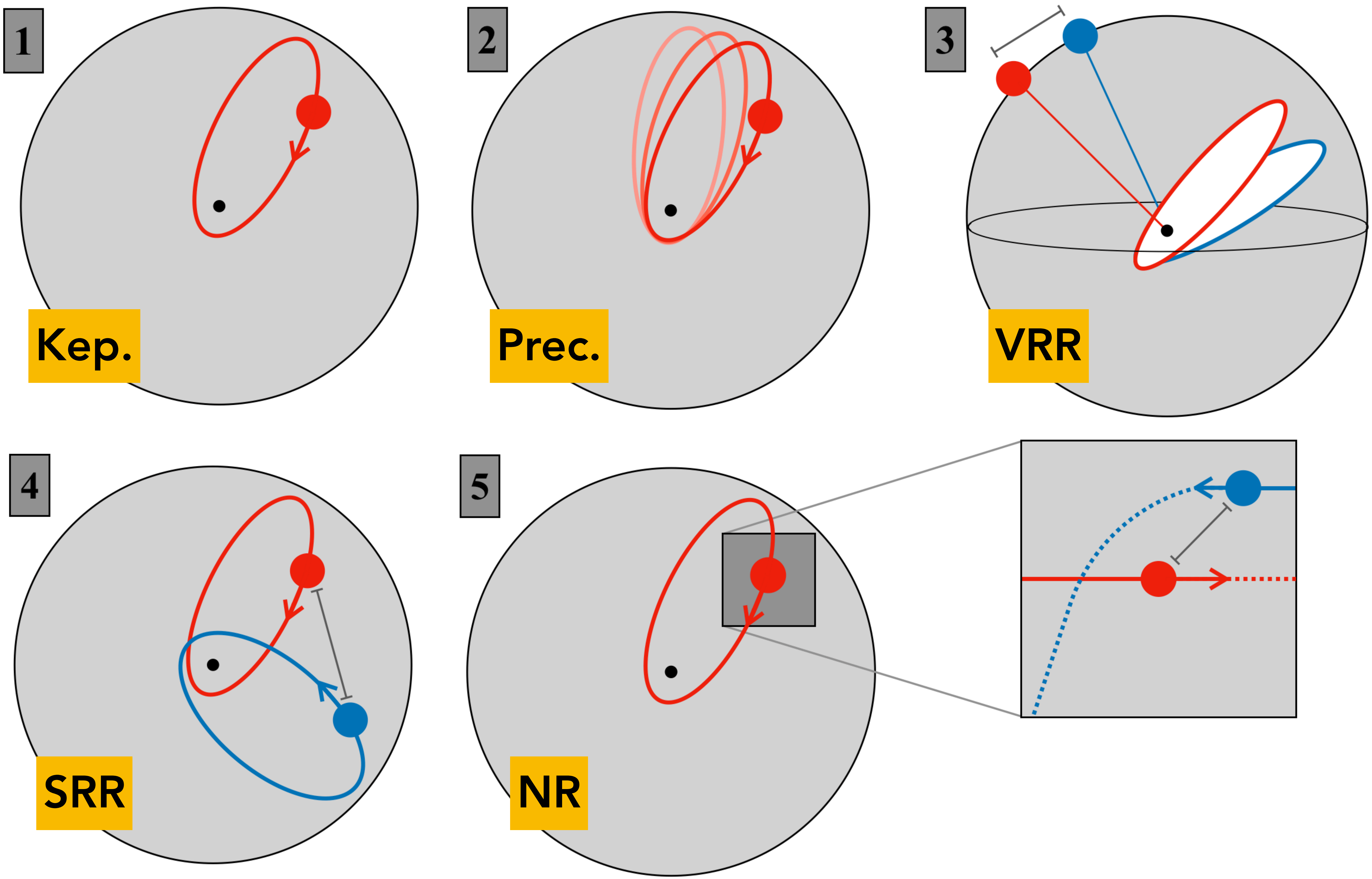
$$\frac{\mathrm{d}e}{\mathrm{d}t} = \eta(e, t)$$

5. Non-Resonant Relaxation
Local two-body encounters

$$\frac{\mathrm{d}a}{\mathrm{d}t} = \eta(a, t)$$

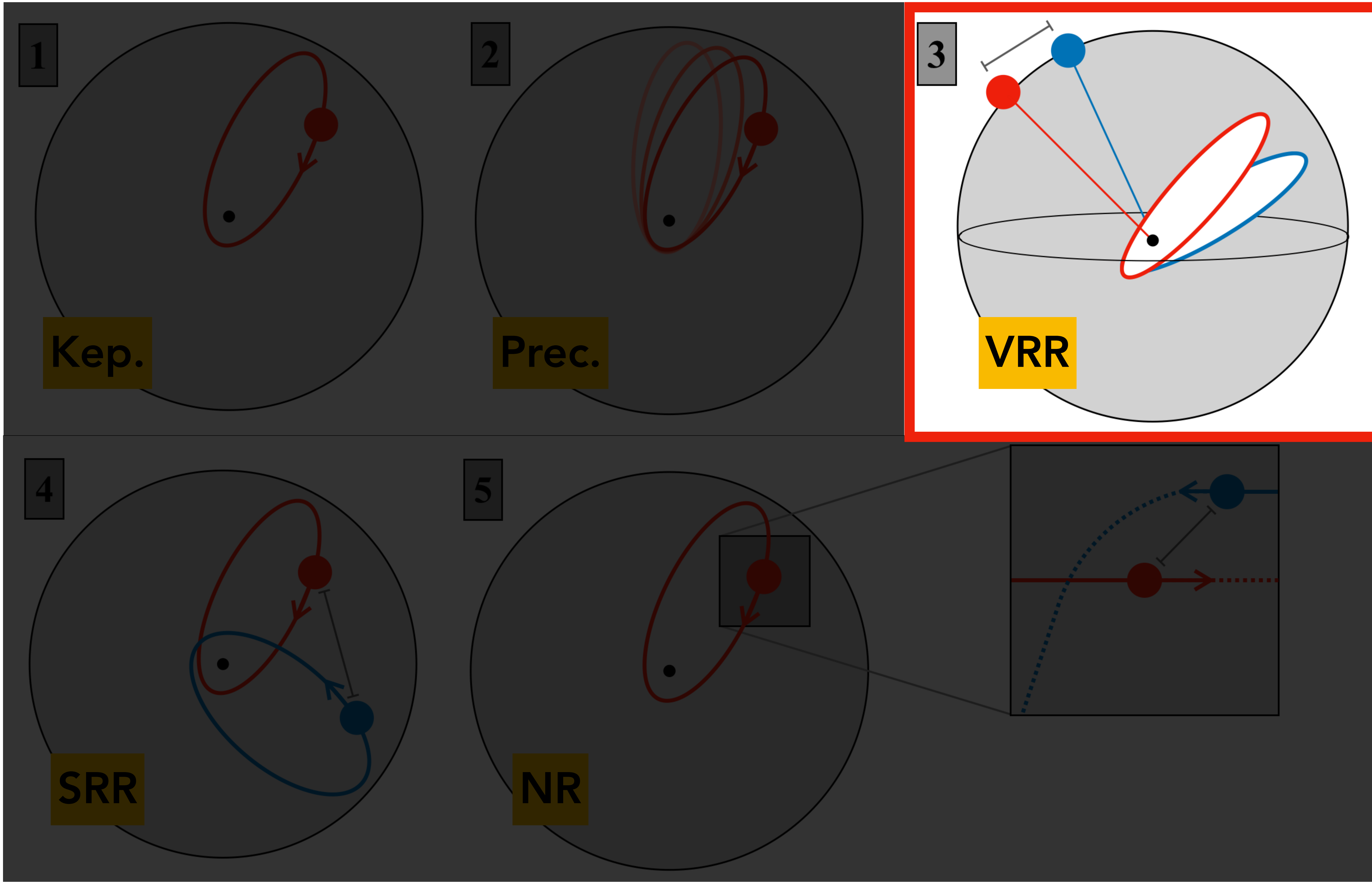


A wealth of dynamical processes



An extremely **hierarchical system**

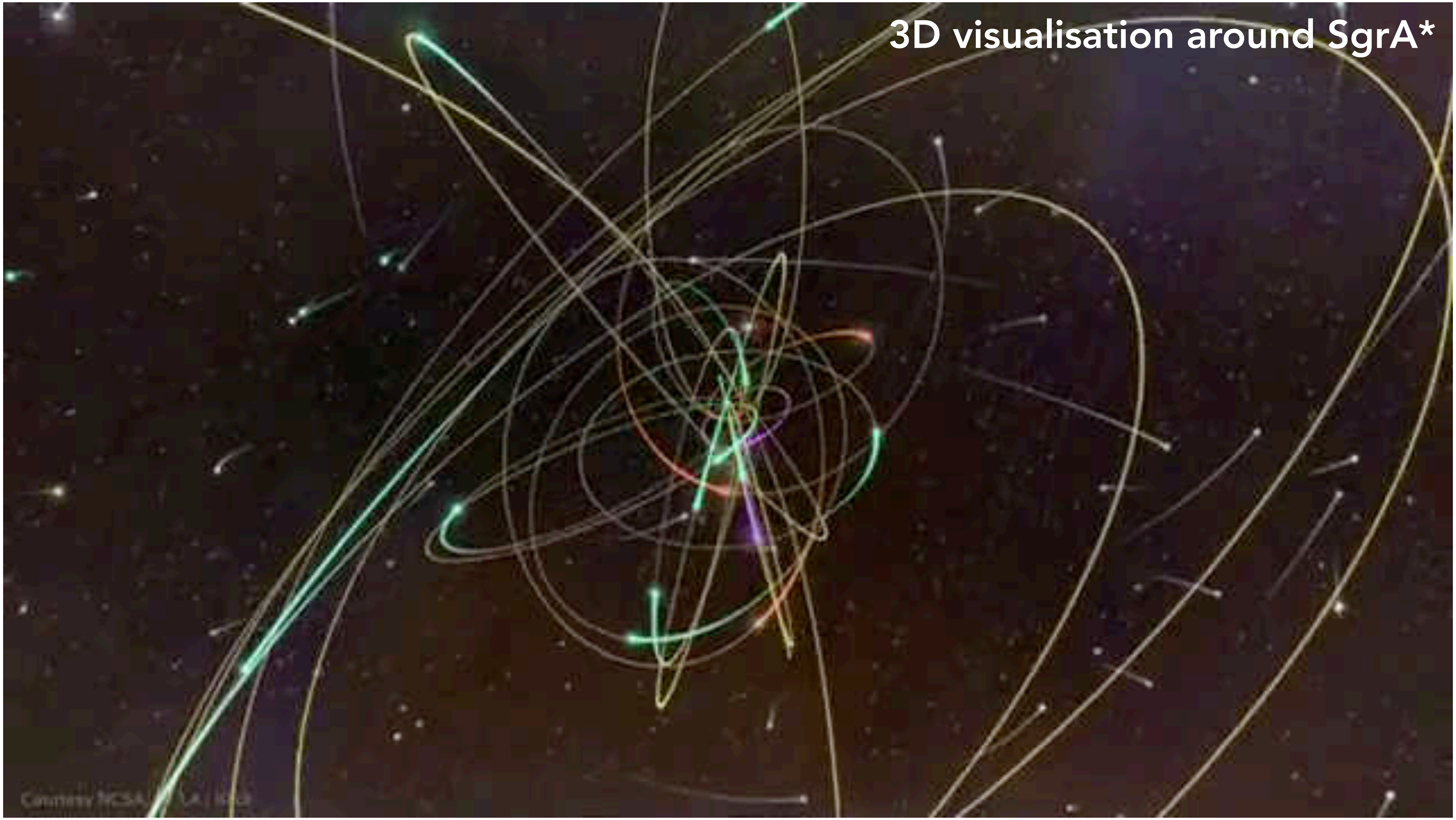
Vector Resonant Relaxation



The coherent dynamics of **orientations**

Stellar orientations

Orbits are in **all directions**

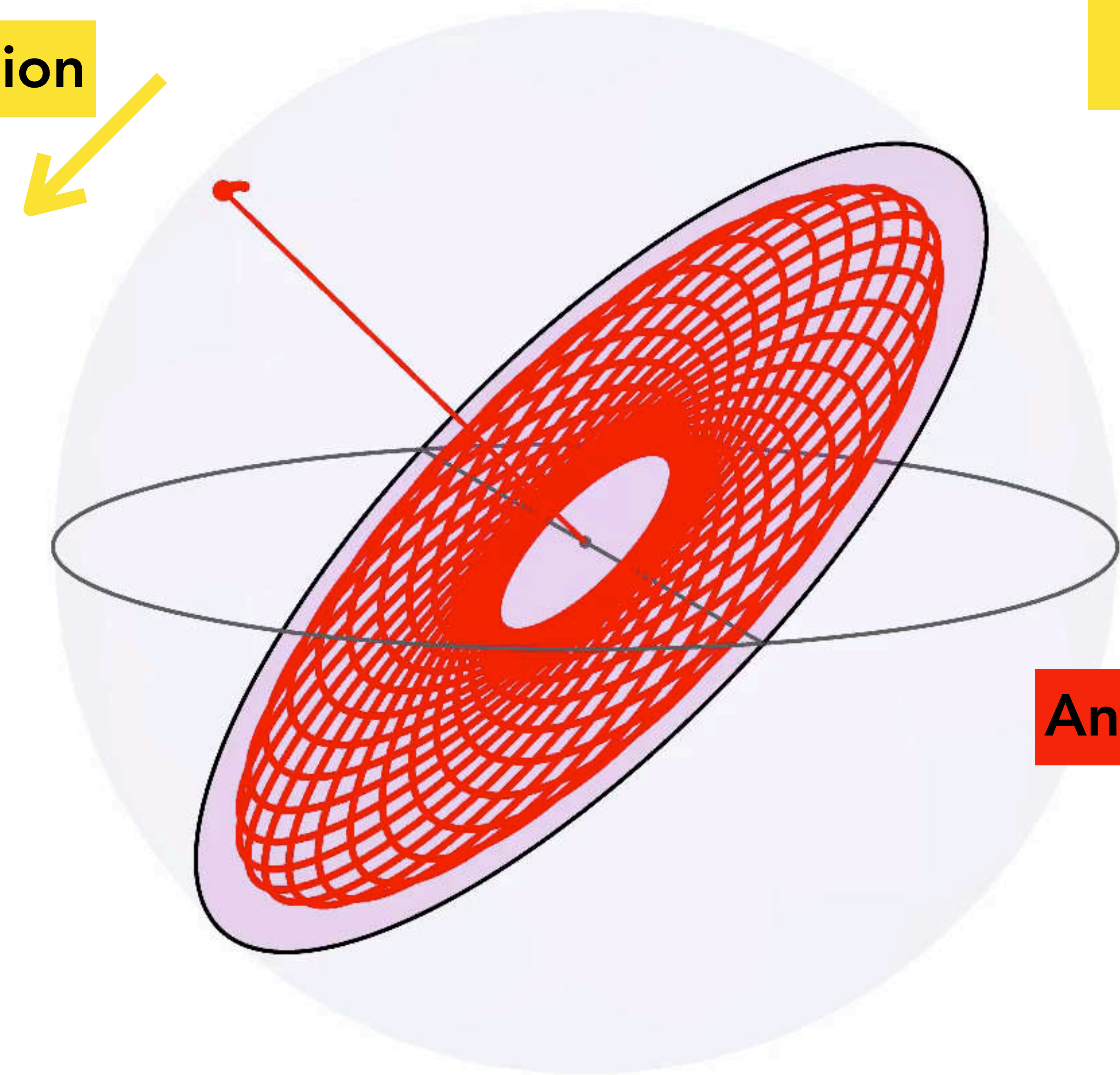


How do stars change of **orientations**?

Stellar orientations

Typical timescale
~1,000,000 years

Orientation

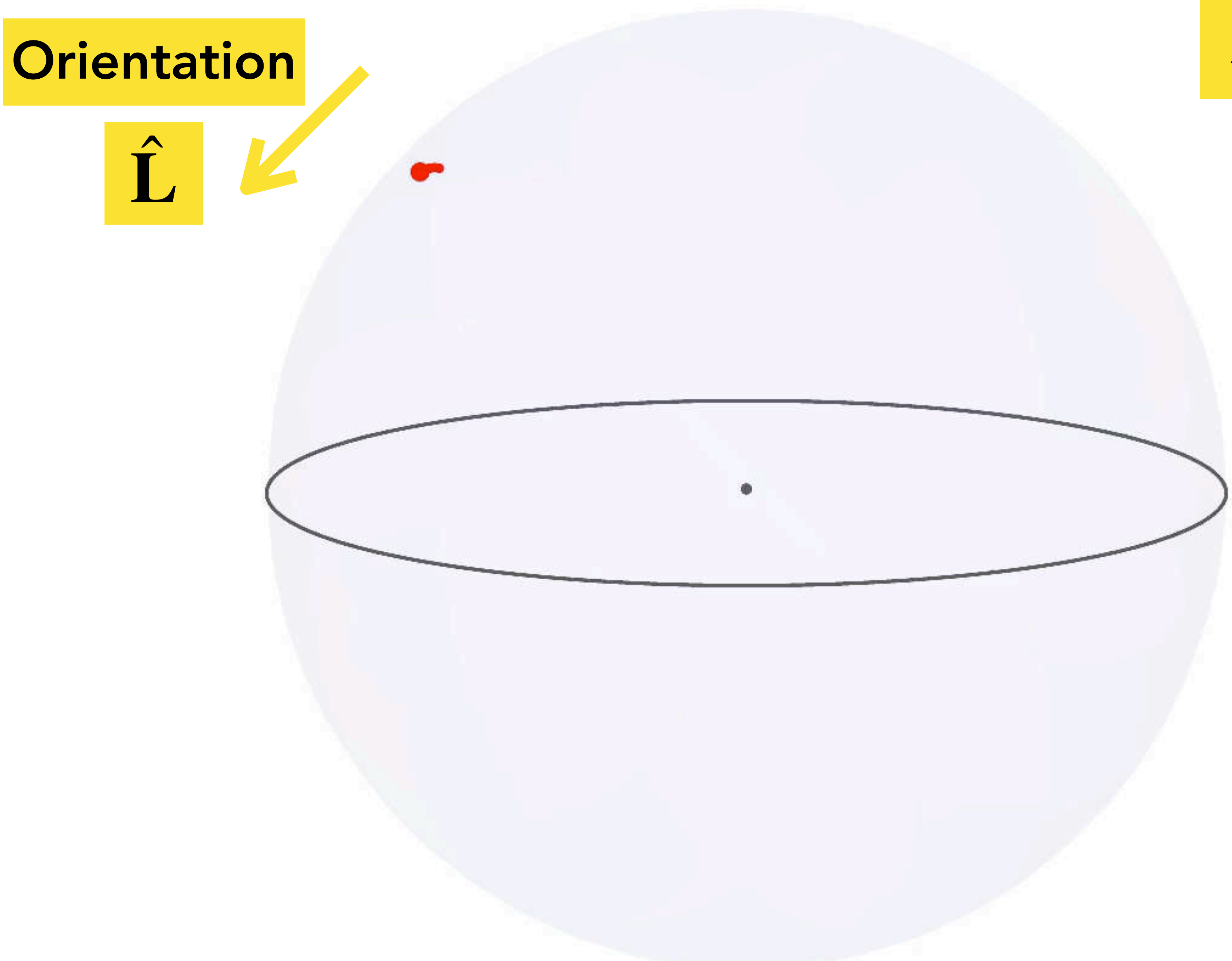


Annuli

After a full precession, **ellipses** become **annuli**

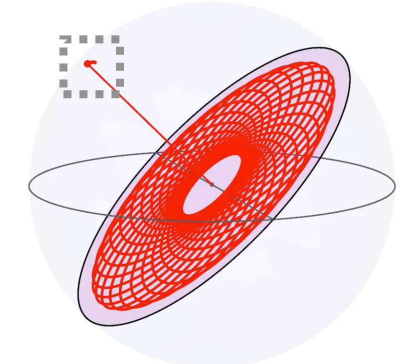
Orbital orientations

Typical timescale
~1,000,000 years



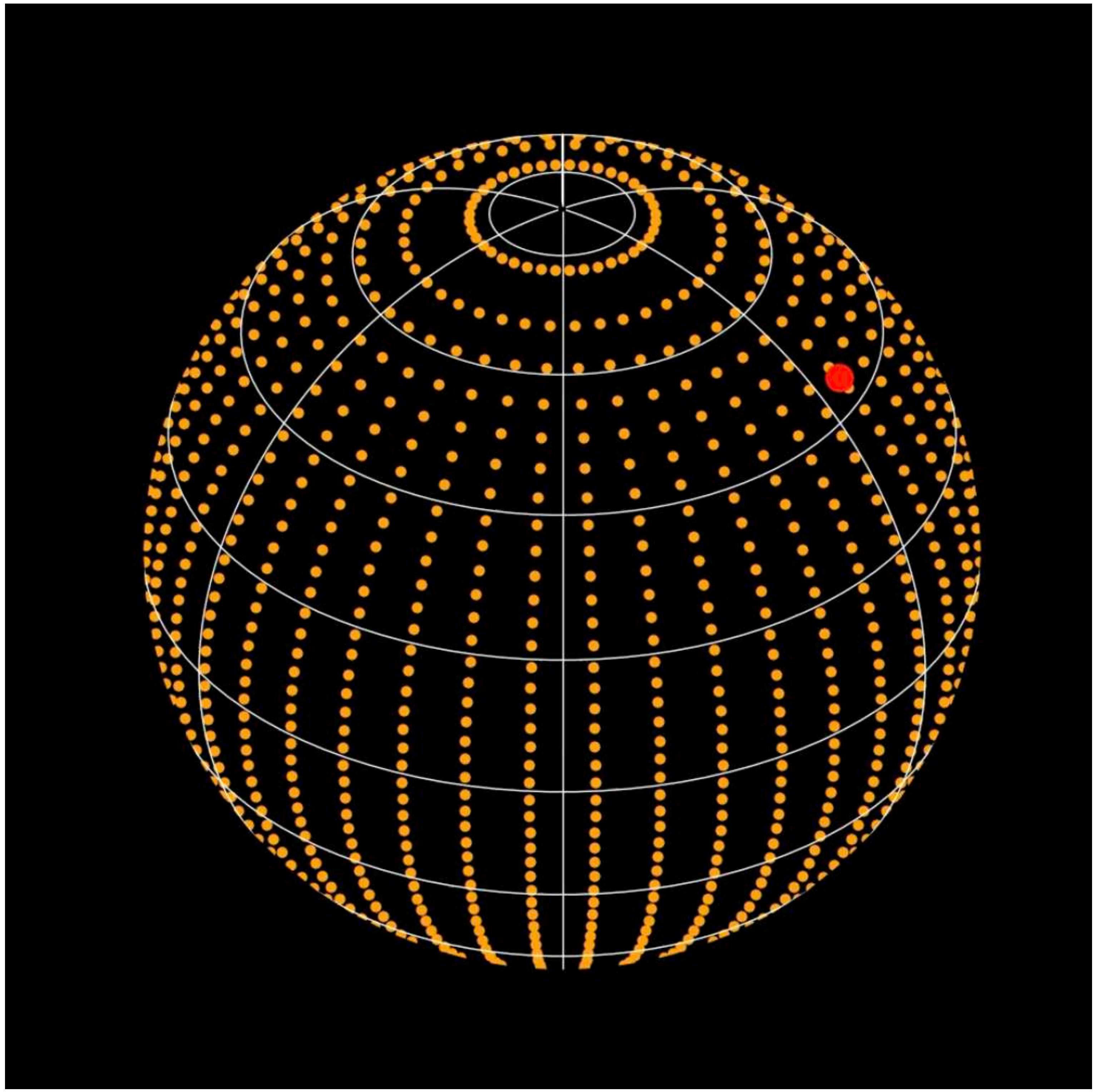
One orientation becomes a single point on the **unit sphere**

Restricted 2-body problem



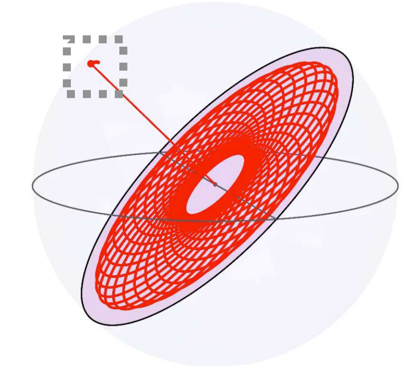
Heavy star

Zero-mass test stars



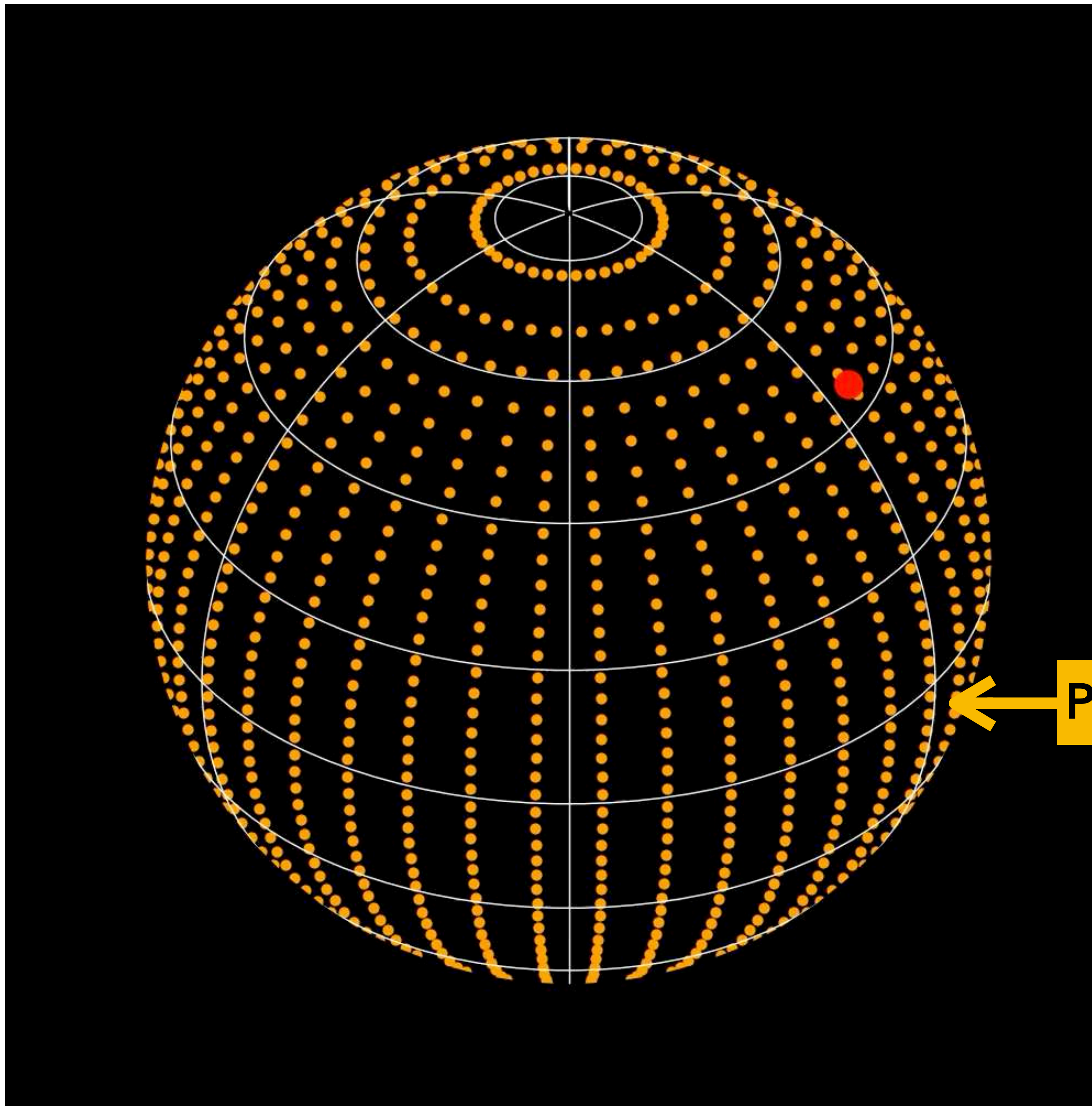
Dynamics induced by a single massive star

Restricted 2-body problem



Heavy star

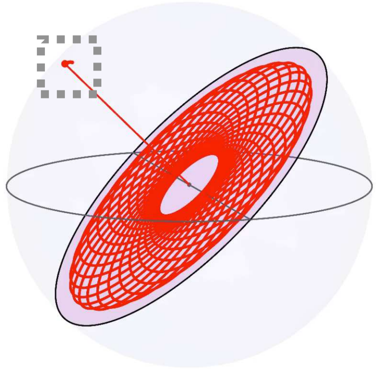
Zero-mass test stars

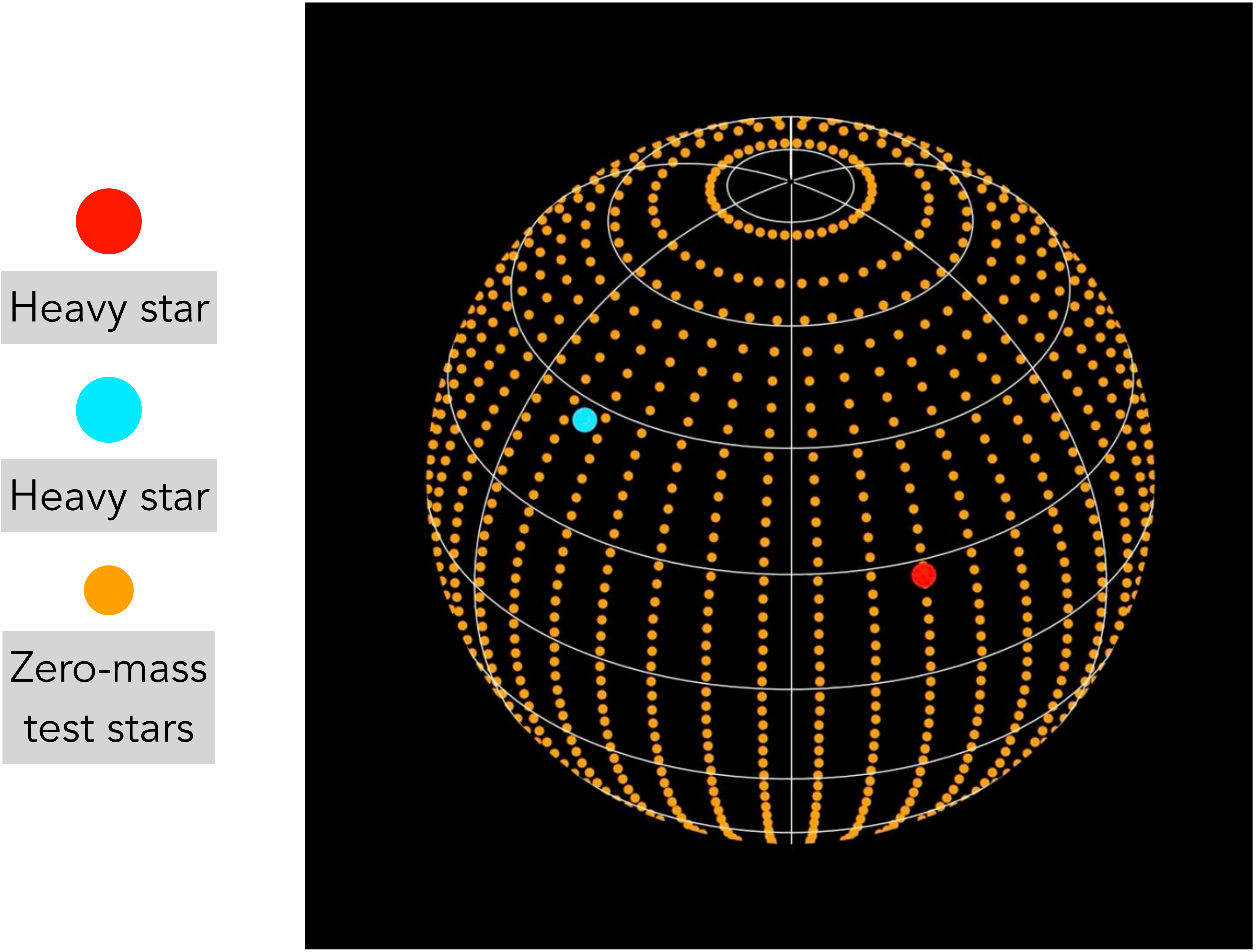


Phase mixing

Simple **orbits** around the massive object

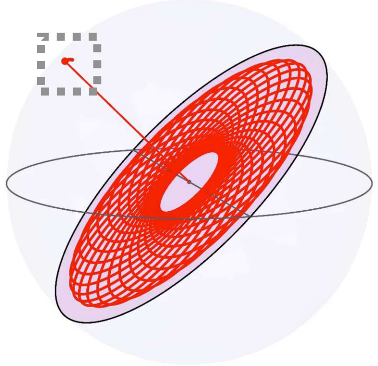
Restricted 3-body problem





Dynamics induced by **two fixed massive stars**

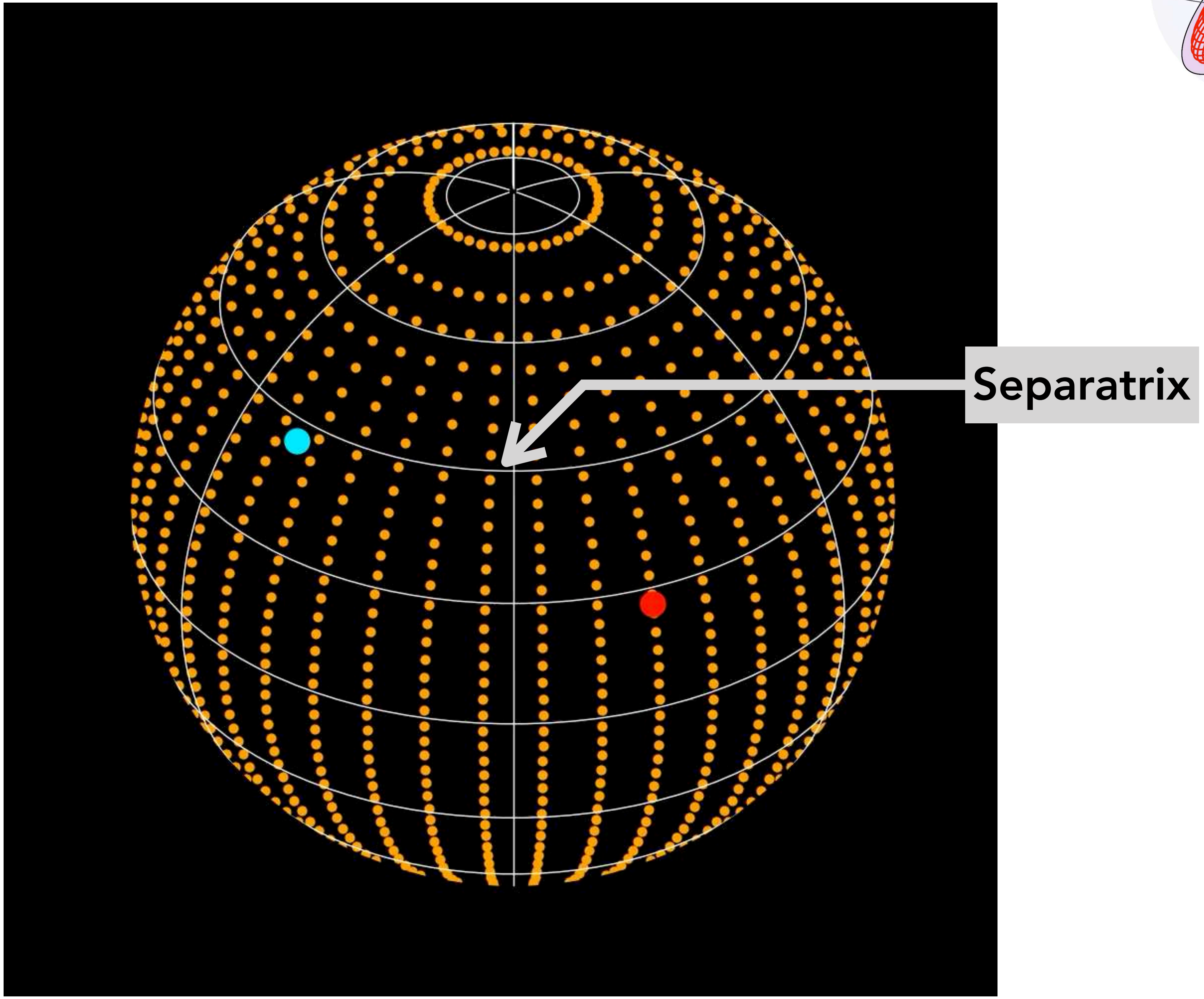
Problème à trois corps restreint



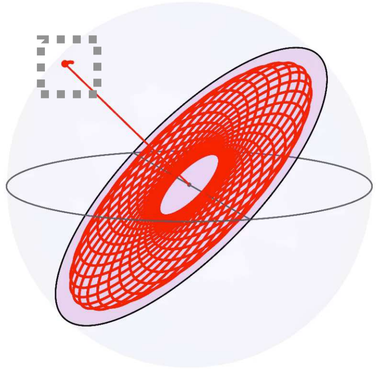
Heavy star

Heavy star

Zero-mass test stars



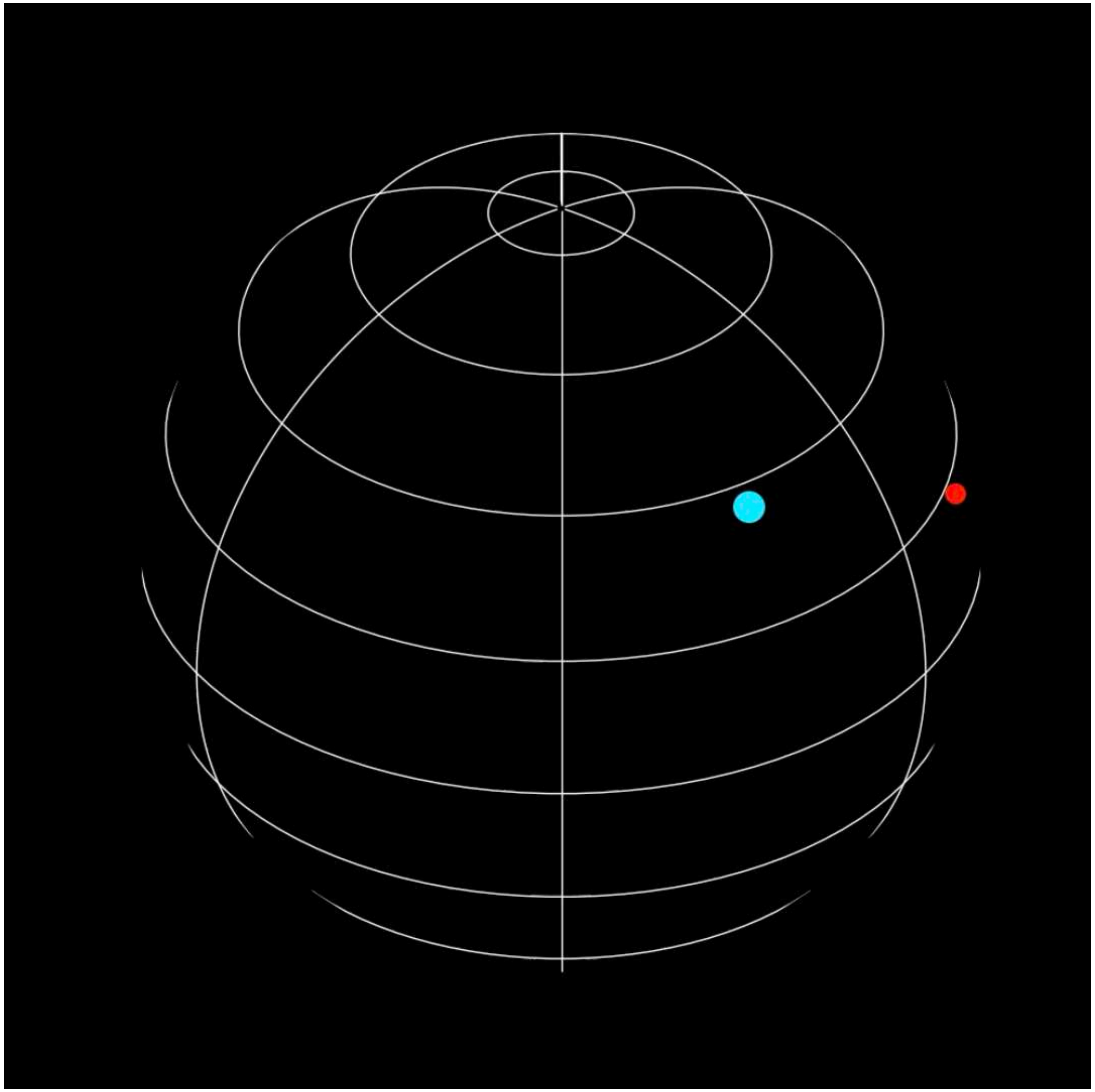
Test stars attracted by **each star respectively**



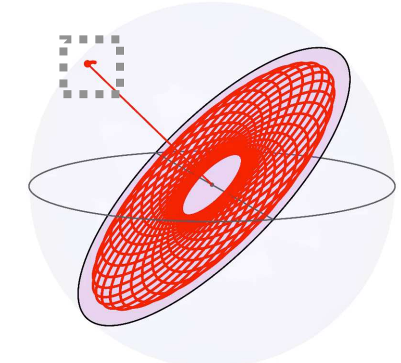
Real 2-body problem

Less
massive star

More
massive star



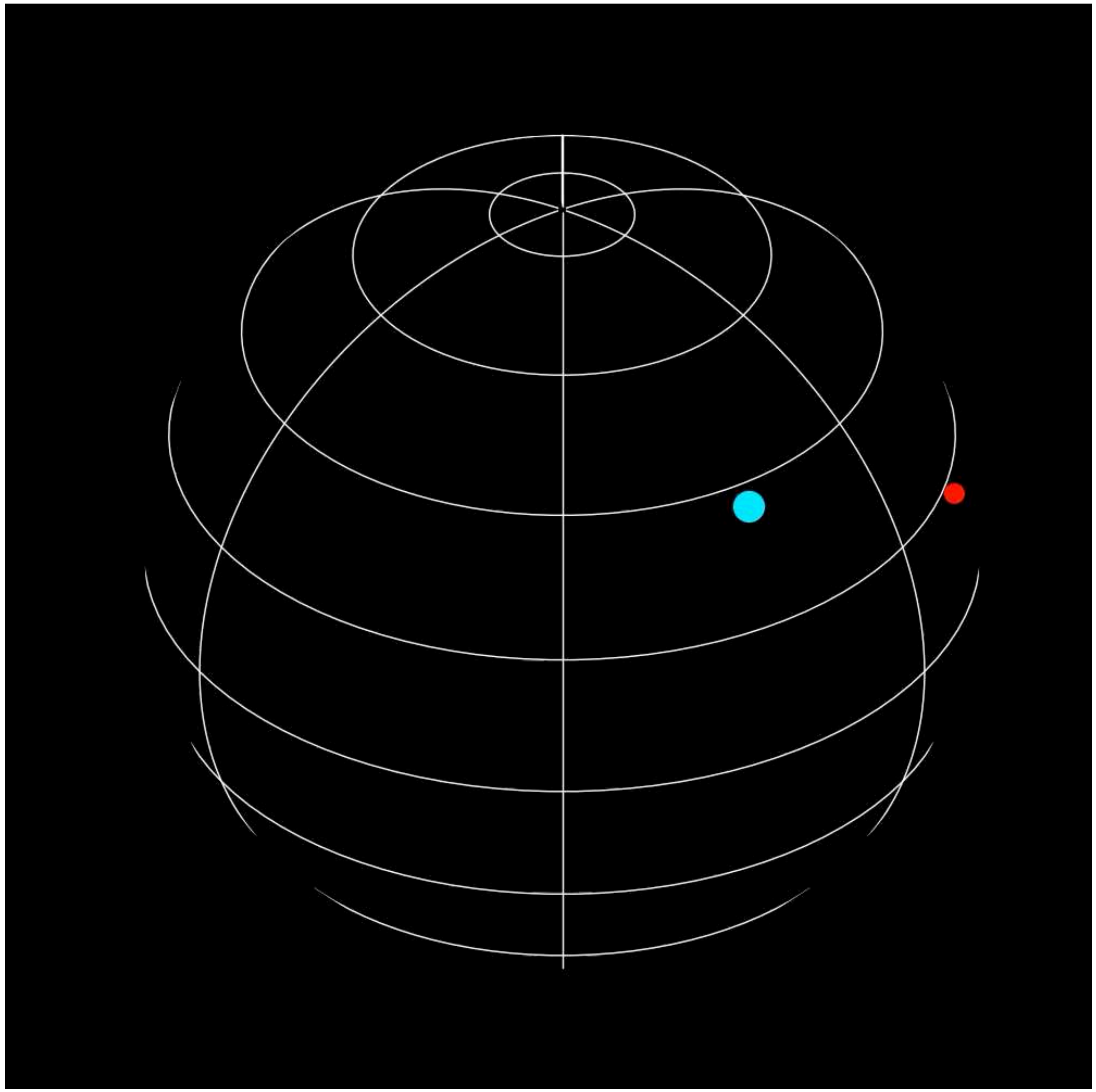
Two massive stars together



Real 2-body problem

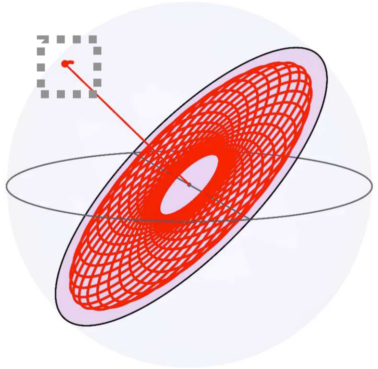
Less
massive star

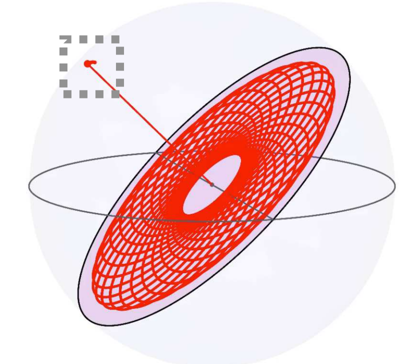
More
massive star



Stars orbit around their common ``**centre of mass**''

3-body problem



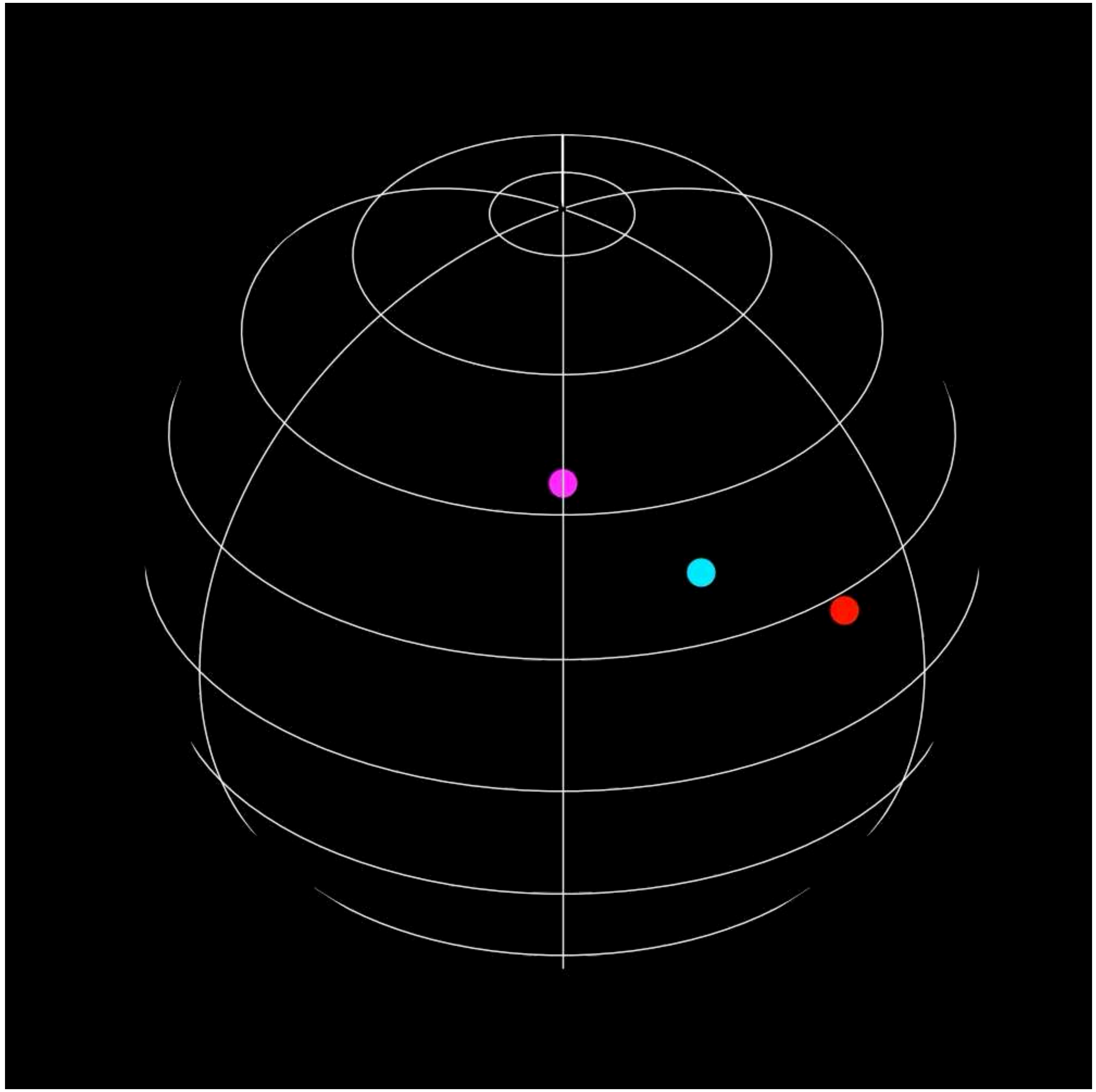


3-body problem

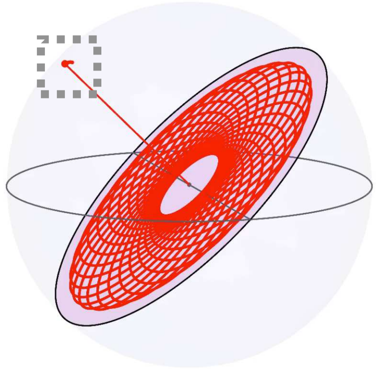
Heavy star

Heavy star

Heavy star



Dynamics is **integrable!**



Heavy star

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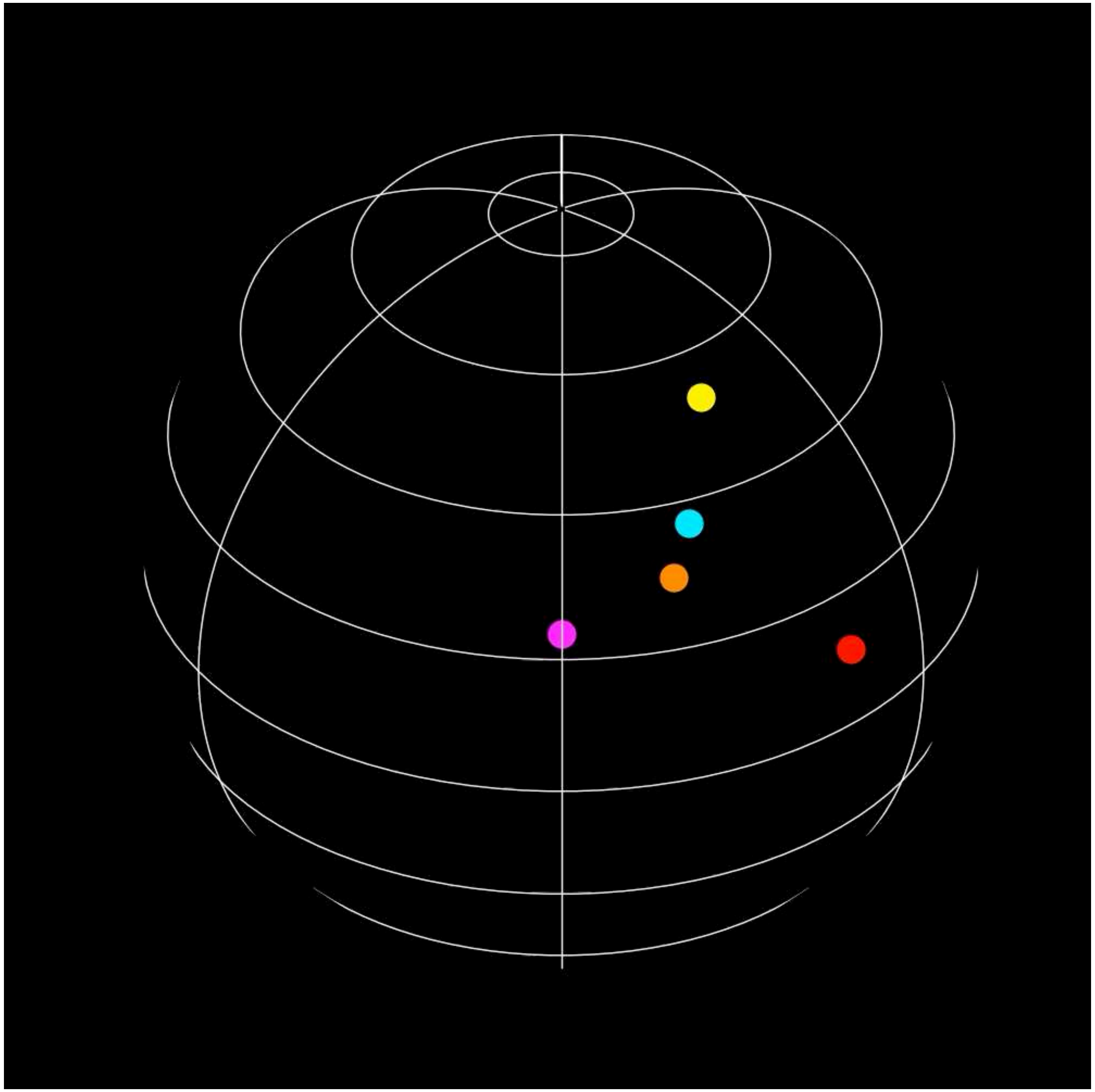
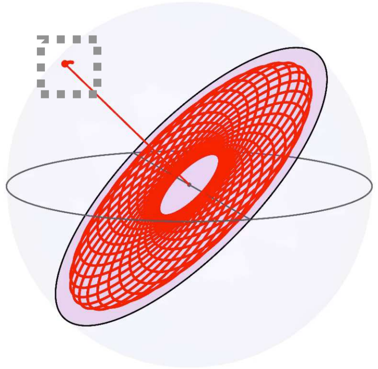


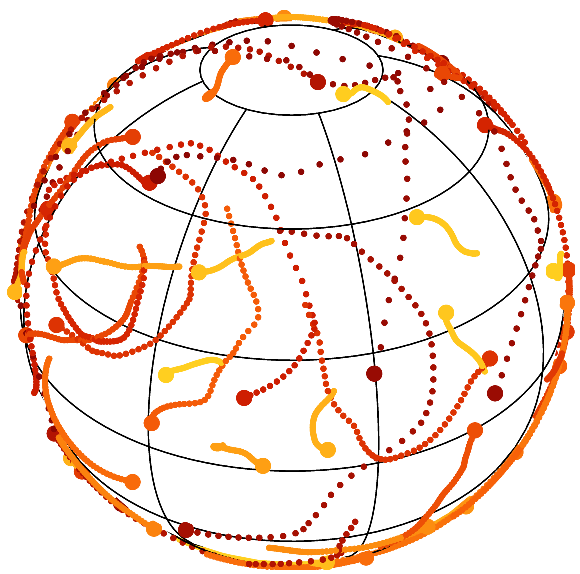
Relaxation starts occuring

A “turbulent” dynamics

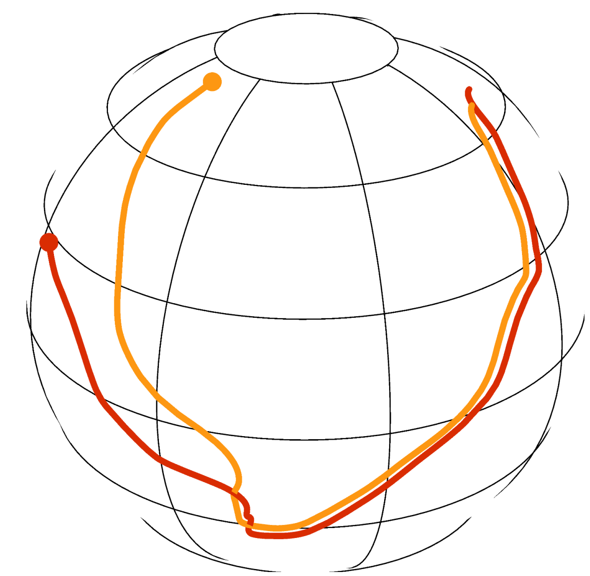


Some aspects of VRR

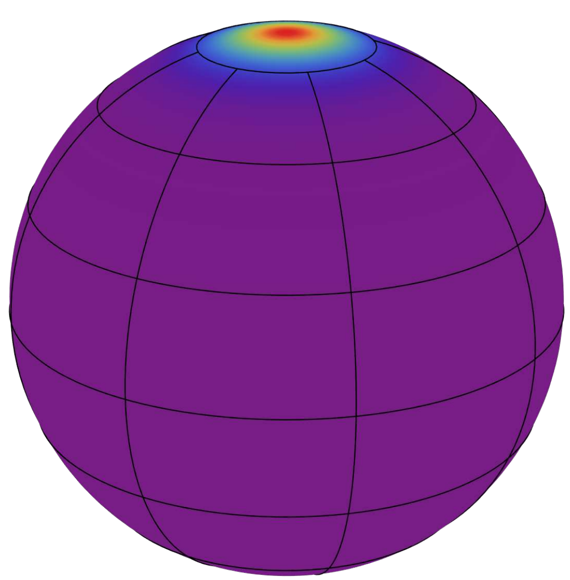
Statistics & Correlation



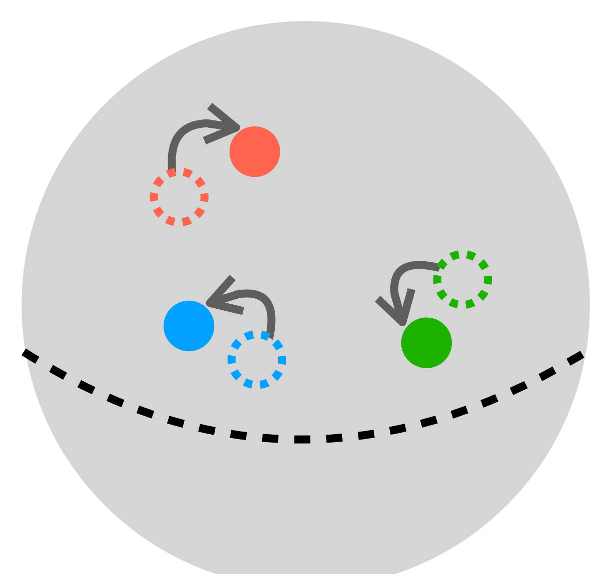
Diffusion & Dilution



Thermodynamics

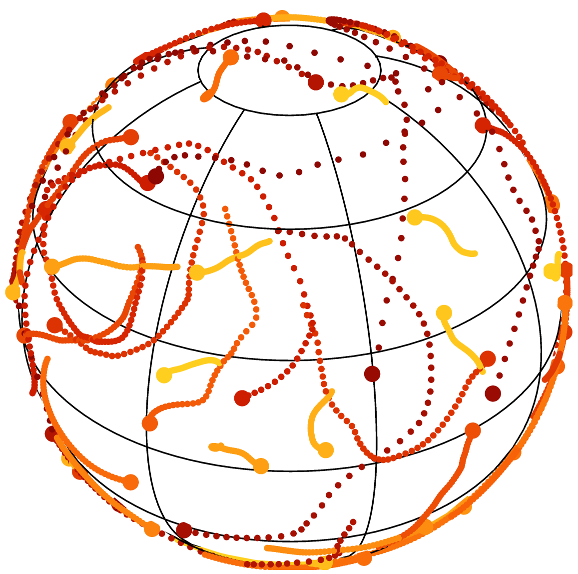


Numerical integration



Some aspects of VRR

Statistics & Correlation



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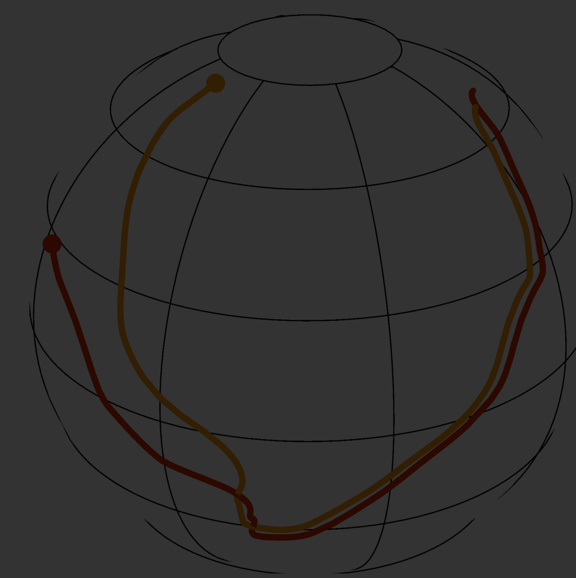

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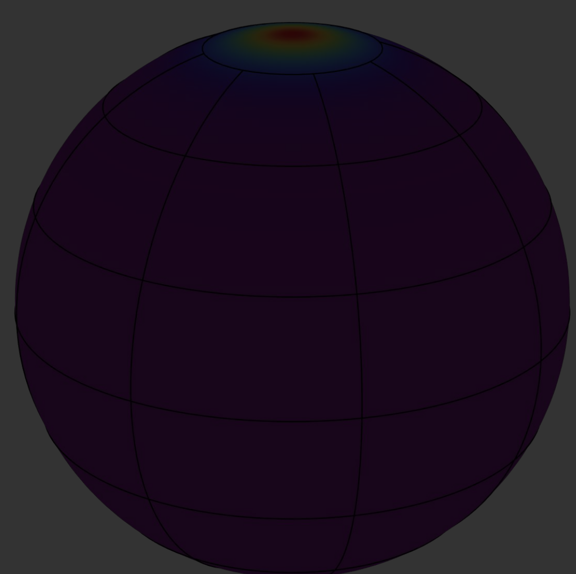
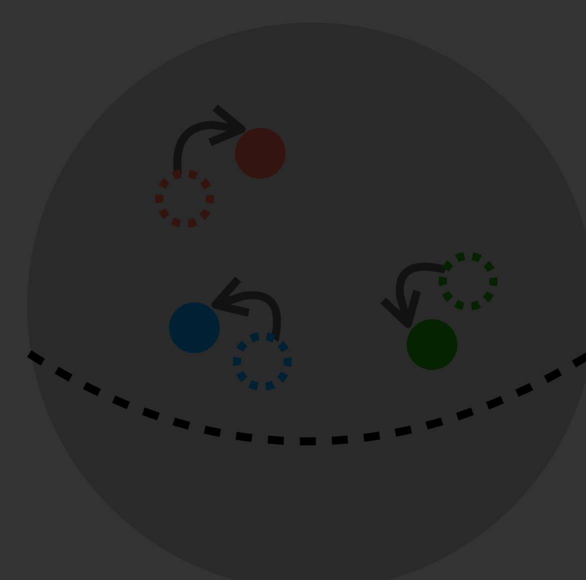
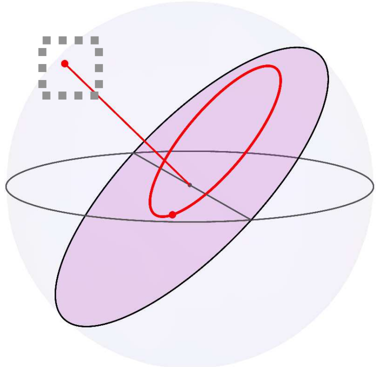
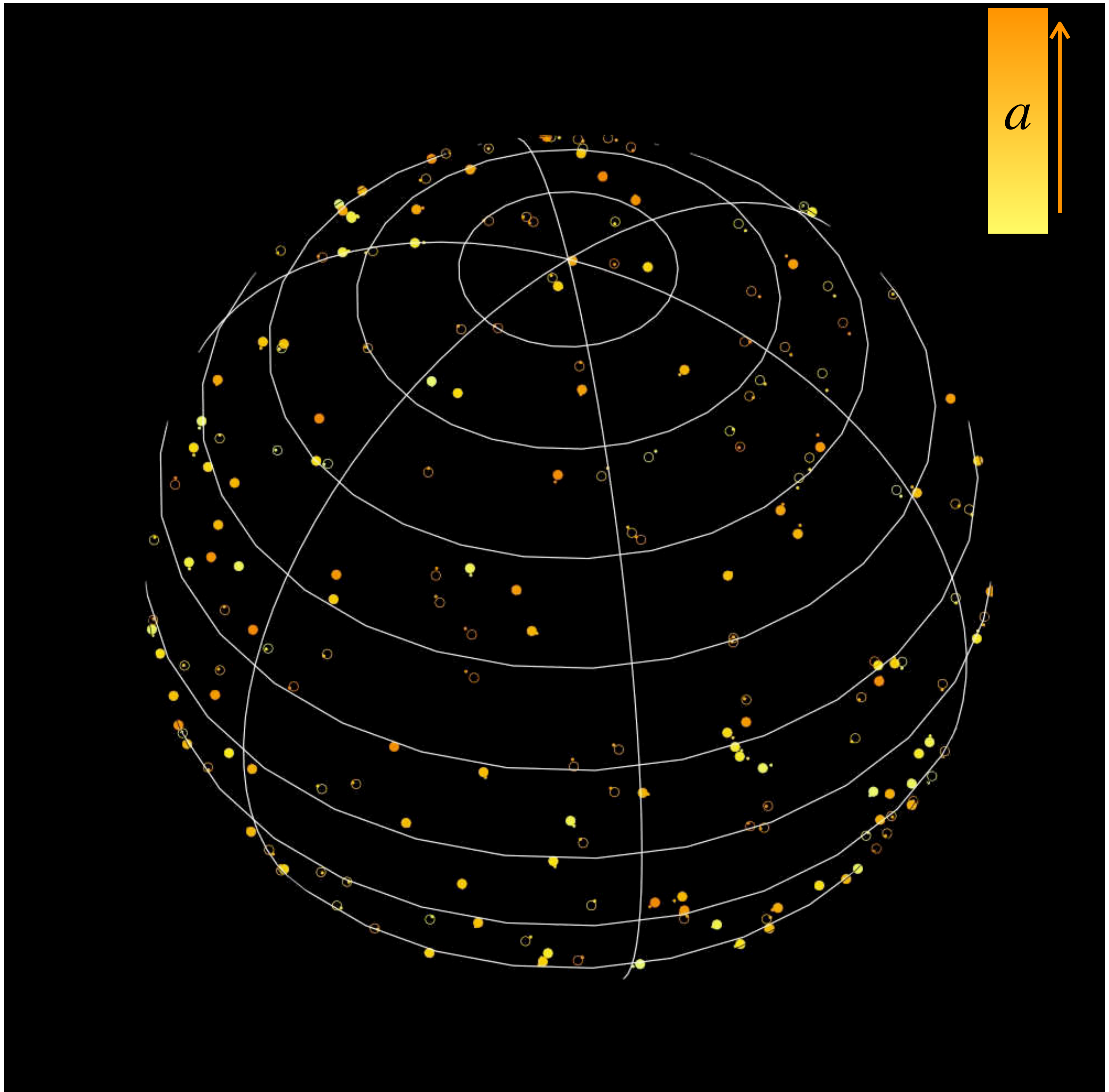


Numerical integration



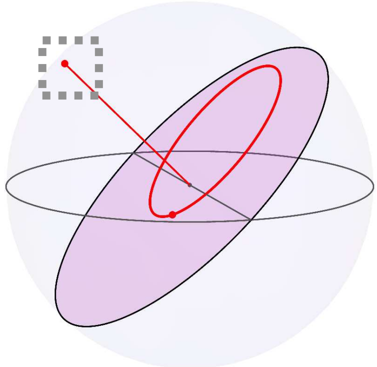
Vector Resonant Relaxation

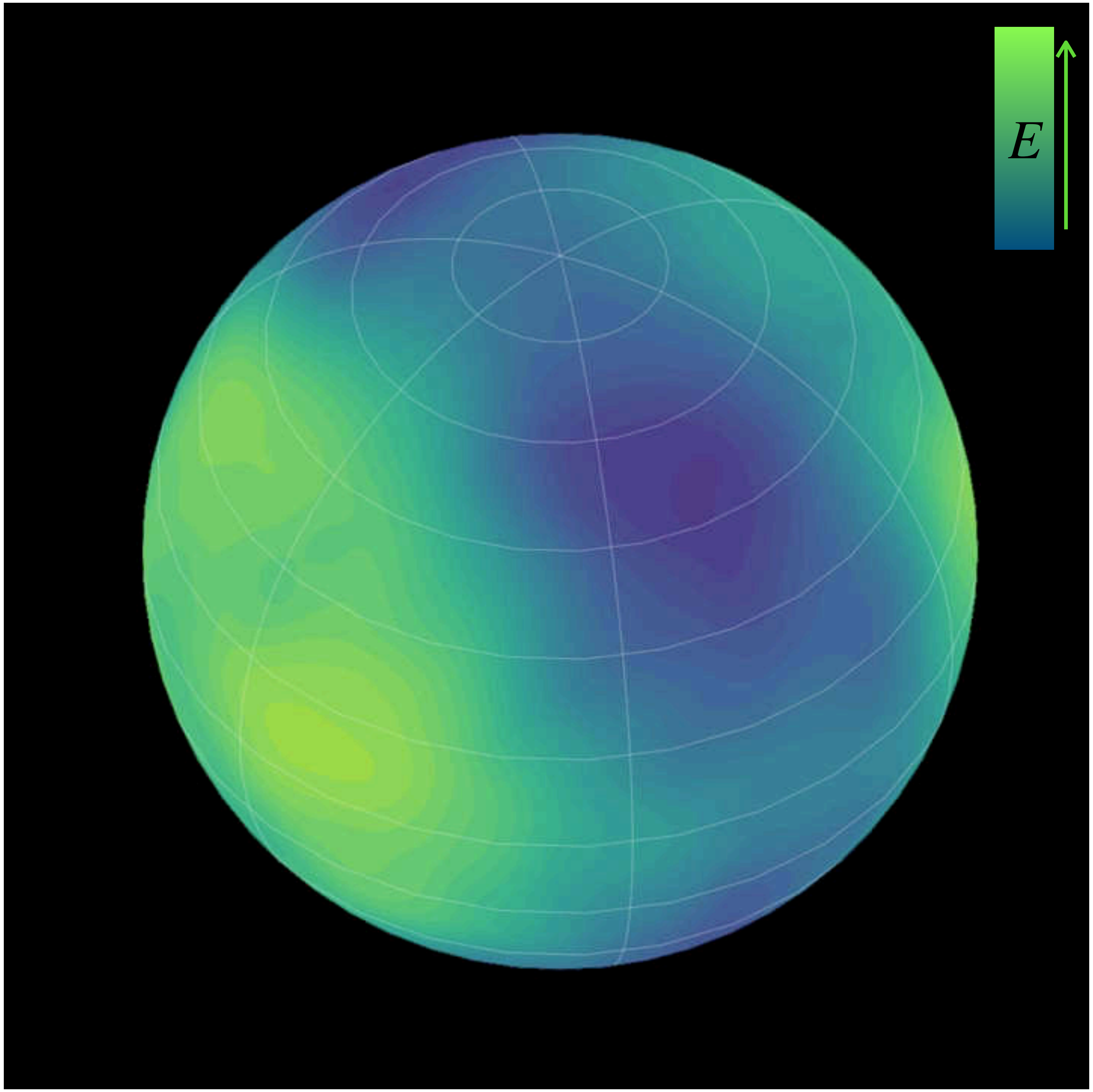




+ Motion coherent on large scales
    - **Long-range interacting system**
+ Motion smooth on short times
    - **Time-correlated noise**
+ Particles have ``preferred friends''
    - **Parametric coupling** (a, e)
+ System in statistical equilibrium
    - **Time stationarity** $(t - t')$
    - **Rotation invariance** $(\hat{\mathbf{L}} \cdot \hat{\mathbf{L}}')$

Vector Resonant Relaxation





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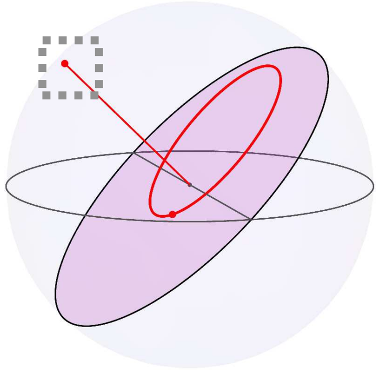
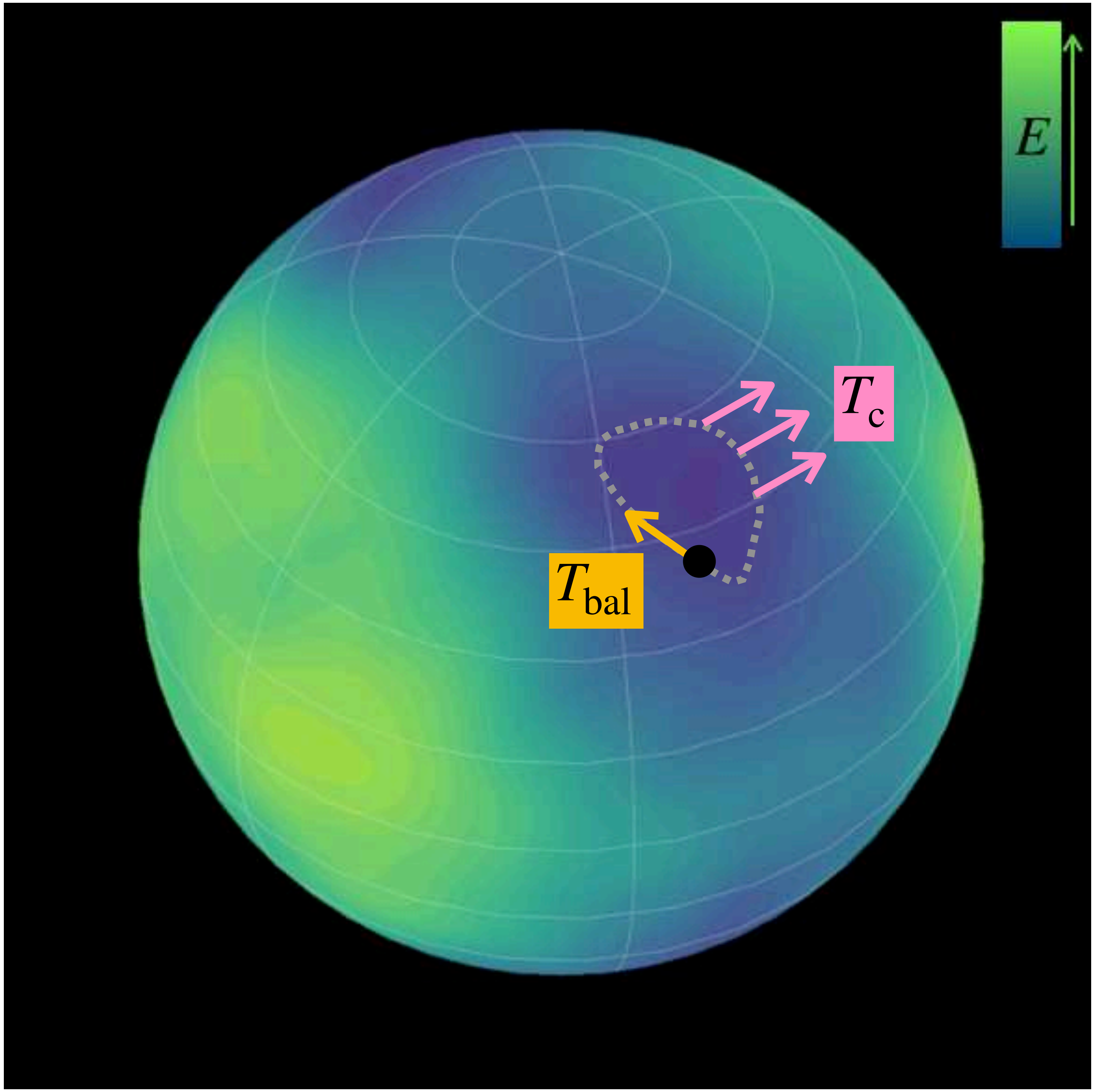
Vector Resonant Relaxation





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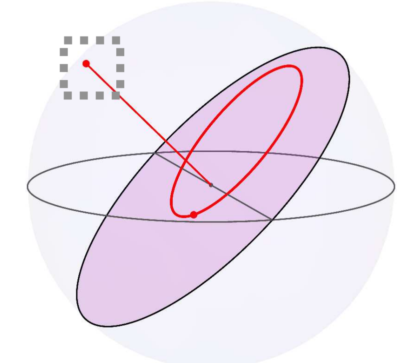
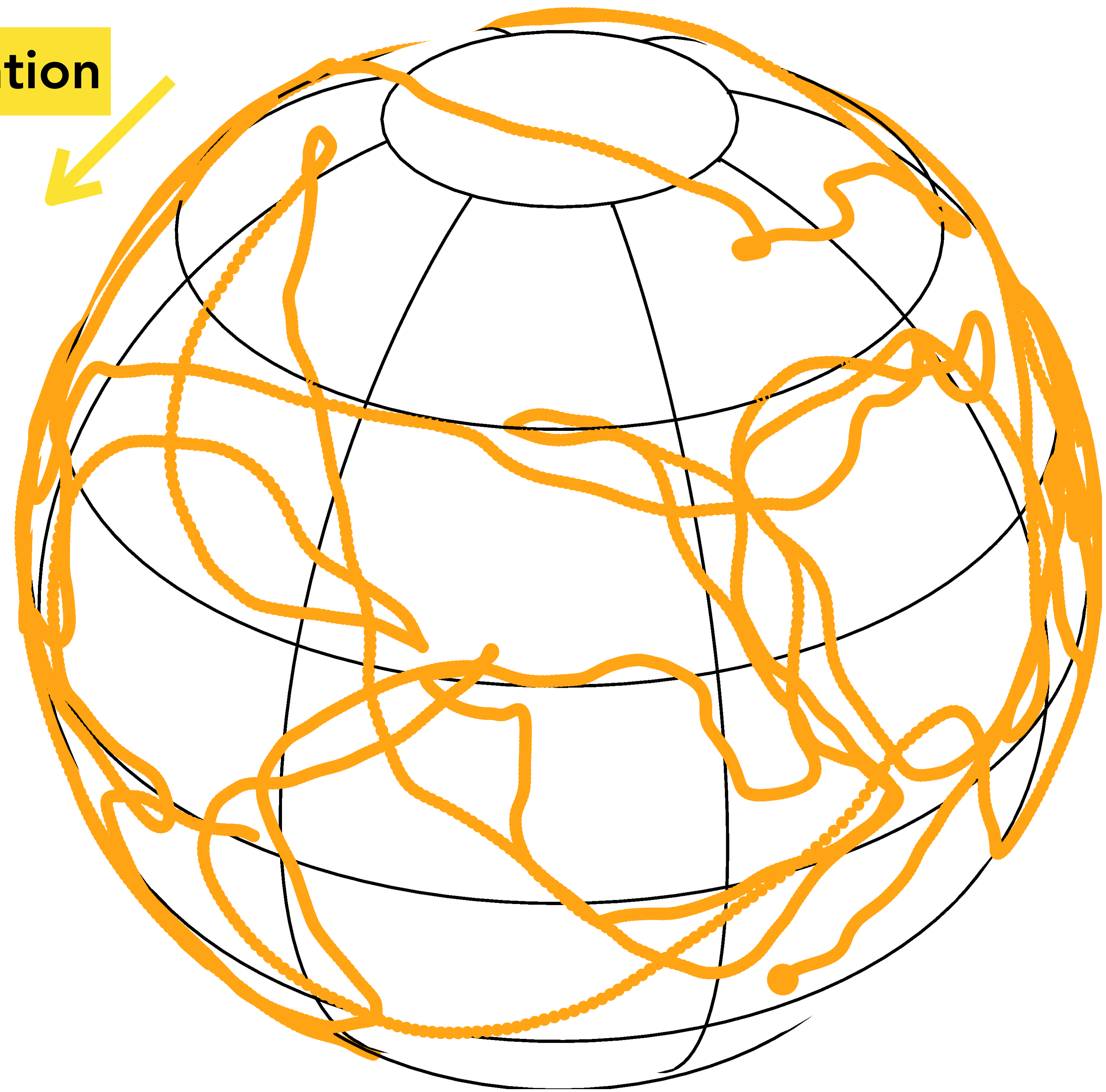
Random walk on the unit sphere



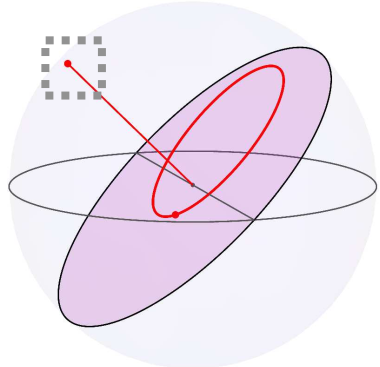
Orientation

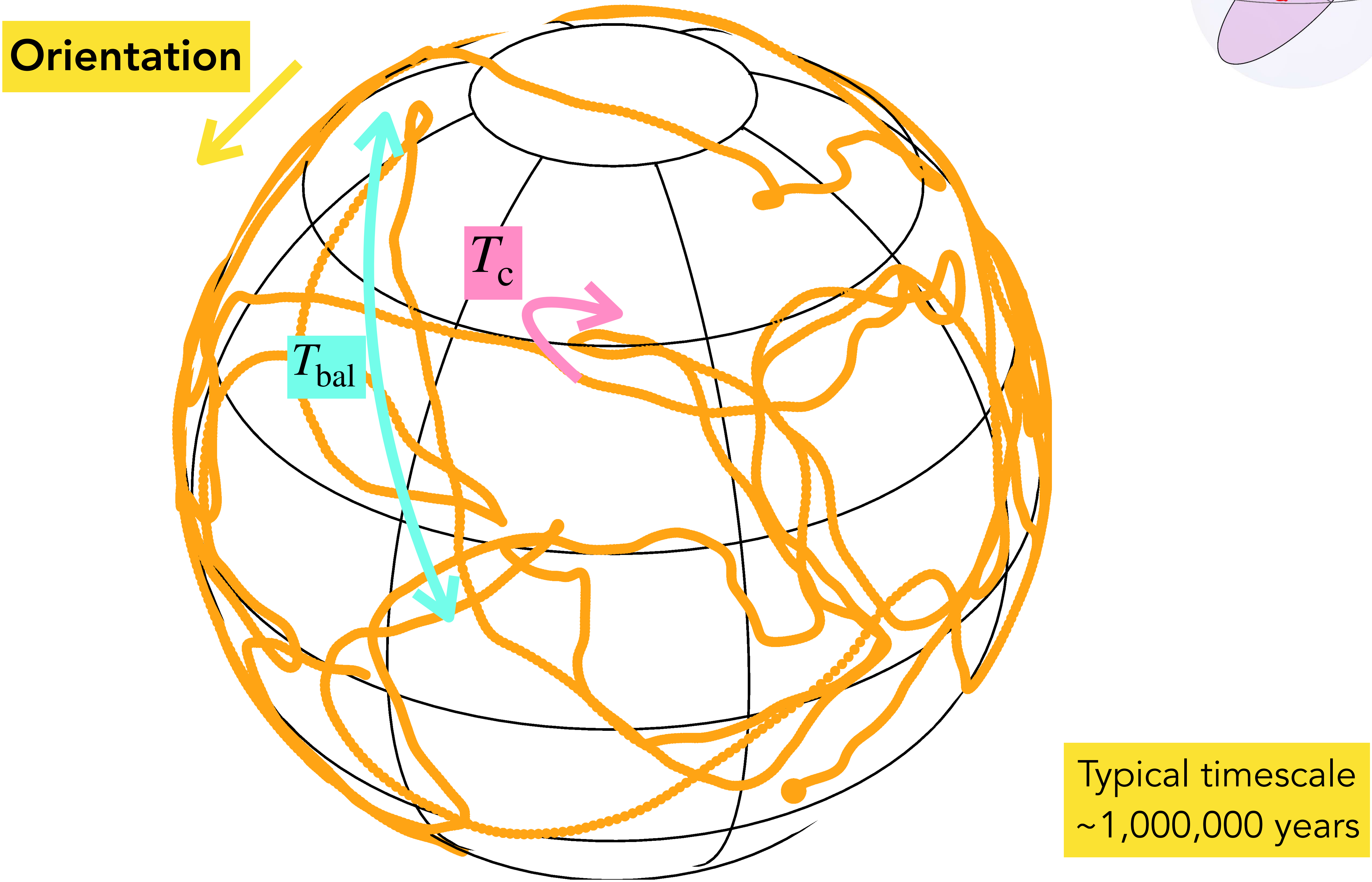


Typical timescale
~1,000,000 years

Stellar orientations follow a **correlated random walk**

Random walk on the unit sphere

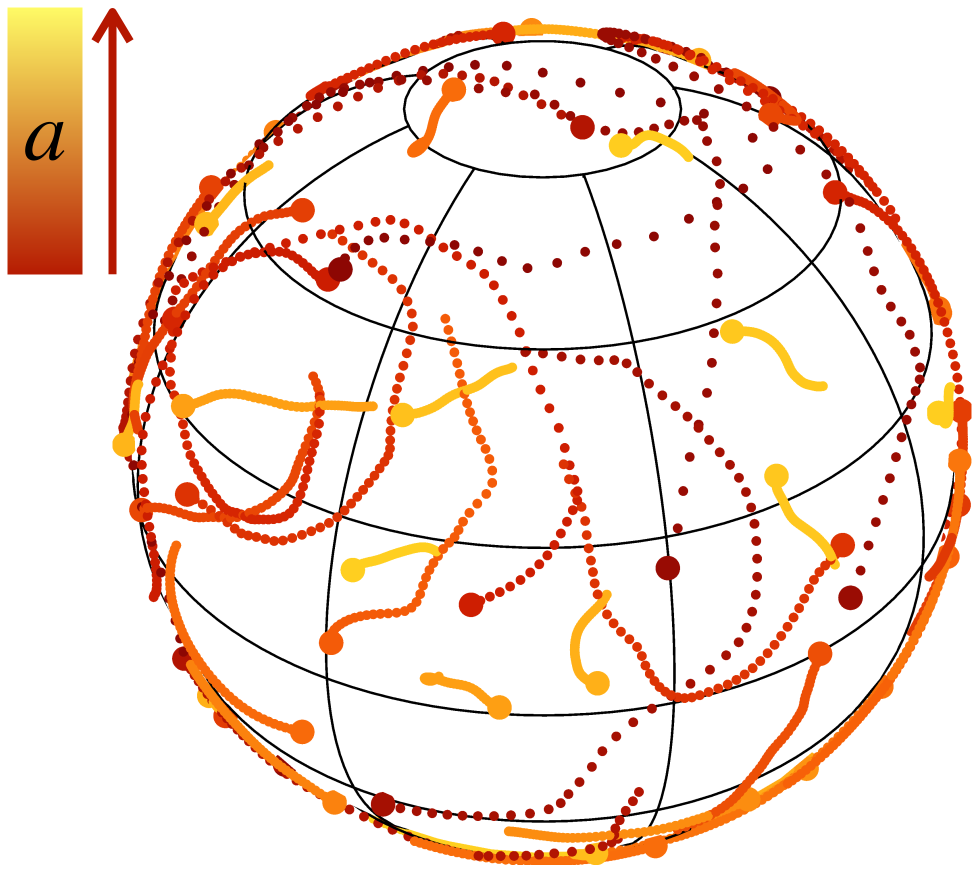




Stellar orientations follow a **correlated random walk**

Self-consistency requirement

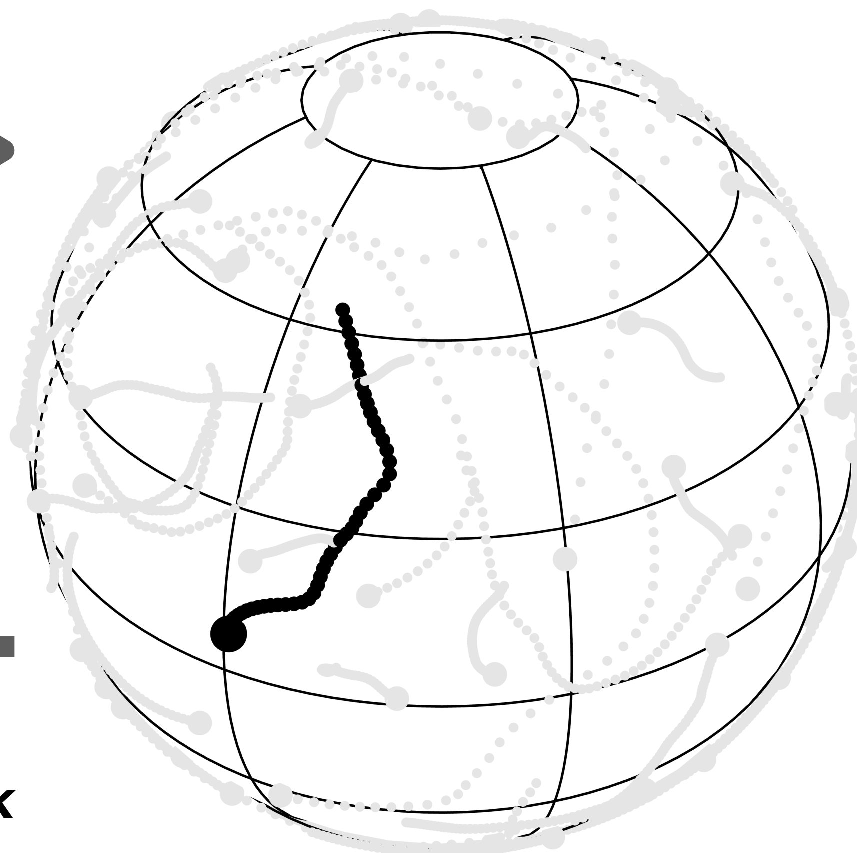
Background particles



Imposes a noisy (correlated) ***potential***

Undergoes a (correlated) ***random walk***

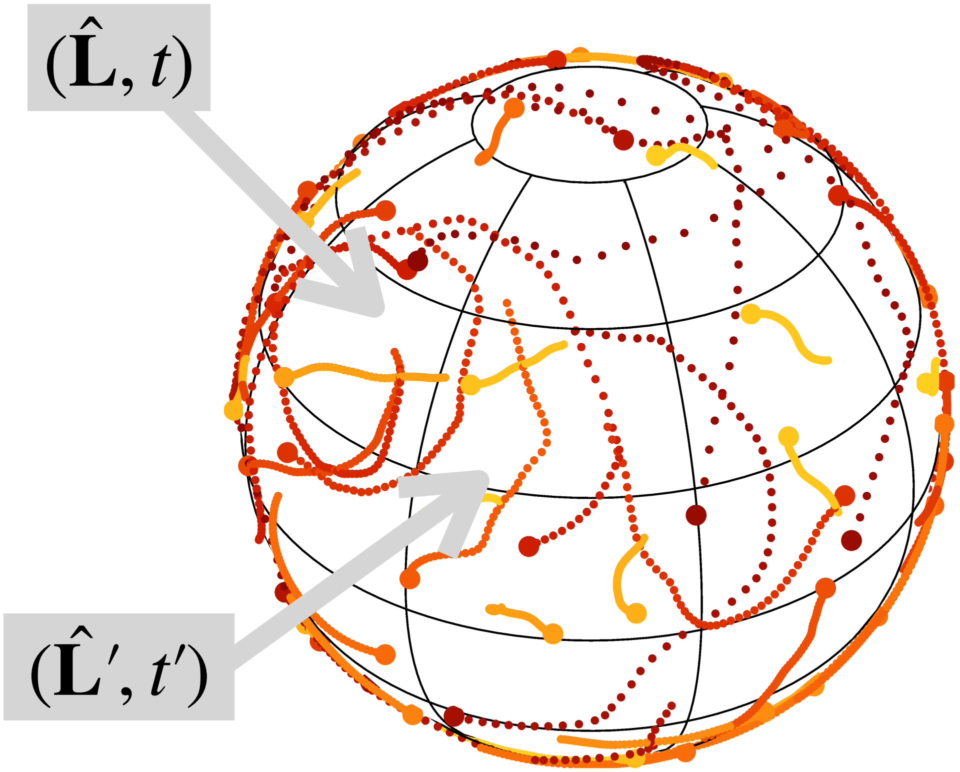
Test particle

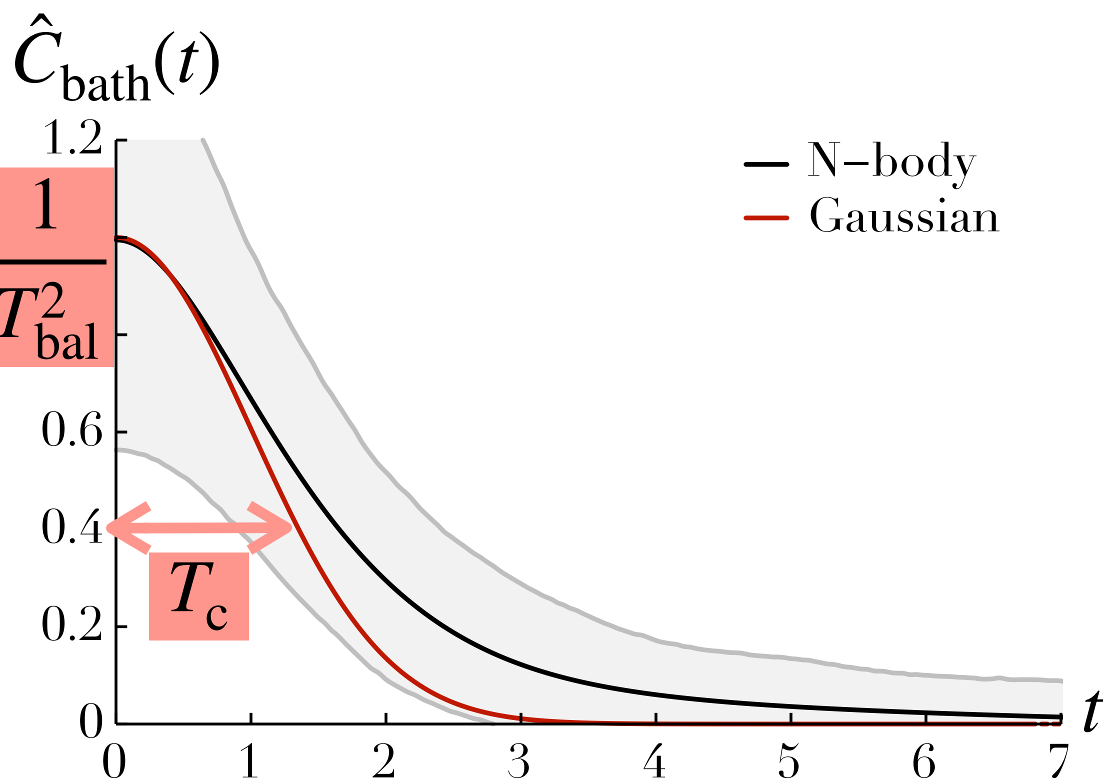


$$\hat{C}_{\mathrm{bath}} = \left\langle \eta(\hat{\mathbf{L}}, t)\, \eta(\hat{\mathbf{L}}', t') \right\rangle$$

$$\hat{C}_{\mathrm{test}} = \left\langle \hat{\mathbf{L}}_{\mathrm{test}}(t) \cdot \hat{\mathbf{L}}_{\mathrm{test}}(0) \right\rangle$$

Characterising the bath noise $\hat{C}_{\mathrm{bath}} = \left\langle \eta(\hat{\mathbf{L}}, t)\, \eta(\hat{\mathbf{L}}', t') \right\rangle$





+ The **state of the bath** is fully characterised by

$$\varphi_{\mathrm{bath}}(\hat{\mathbf{L}}, t) = \frac{1}{N} \sum_{i=1}^{N} \delta_{\mathrm{D}}(\hat{\mathbf{L}} - \hat{\mathbf{L}}_i(t))$$

+ System's (quadratic) **evolution equation**

$$\frac{\partial \varphi_{\mathrm{bath}}(t)}{\partial t} = Q\, \varphi_{\mathrm{bath}}(t)\, \varphi_{\mathrm{bath}}(t)$$

+ Good news
  - At t=0, particles are **statistically decorrelated**
  - Very constraining **spherical symmetries**

+ **Initial time statistics**

$$\left\langle \hat{C}_{\mathrm{bath}}(t=0) \right\rangle$$

Ballistic time

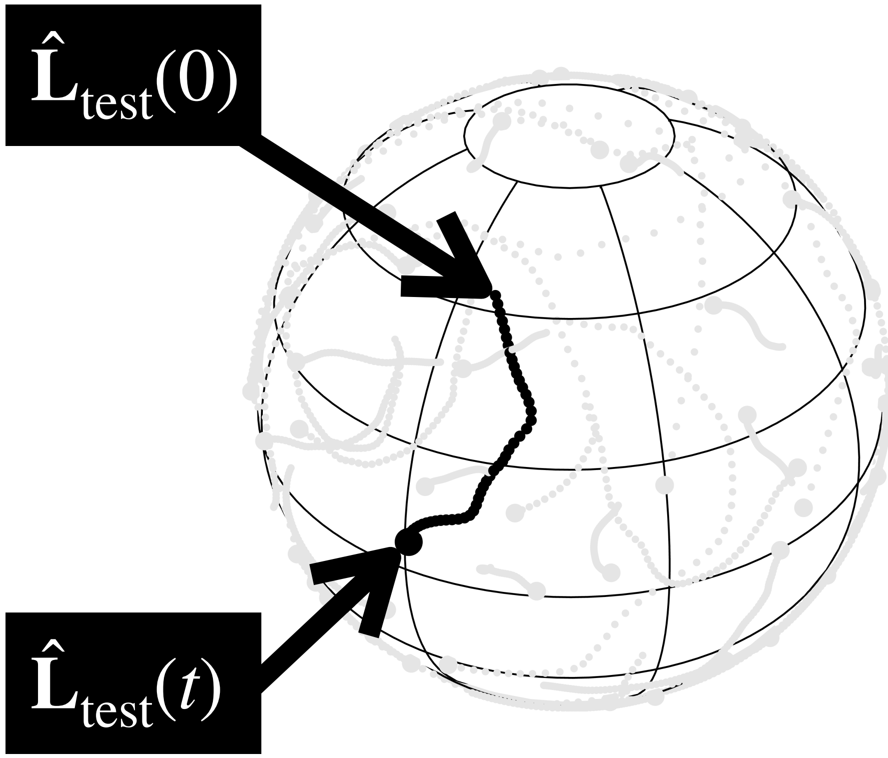
$$\left\langle \left. \frac{\mathrm{d}^2 \hat{C}_{\mathrm{bath}}}{\mathrm{d}t^2} \right|_{t=0} \right\rangle$$

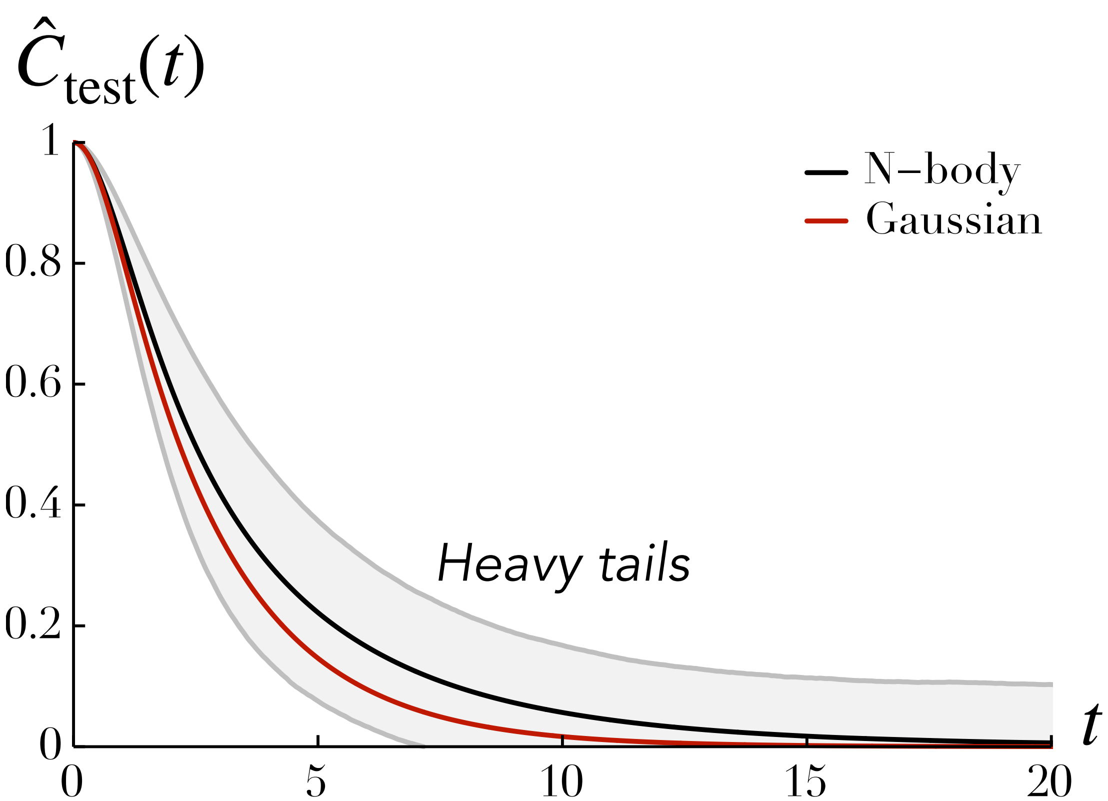
Coherence time

+ (Natural) **Gaussian** Ansatz

$$\hat{C}_{\mathrm{bath}}(t) = \frac{1}{T_{\mathrm{bal}}^2} \mathrm{e}^{-(t/T_{\mathrm{c}})^2}$$

Characterising the random walk $\hat{C}_{\rm test} = \left\langle \hat{\mathbf{L}}_{\rm test}(t) \cdot \hat{\mathbf{L}}_{\rm test}(0) \right\rangle$





+ Location of the **test particle** characterised by

$$\varphi_{\rm test}(\hat{\mathbf{L}}, t) = \delta_{\rm D}(\hat{\mathbf{L}} - \hat{\mathbf{L}}_{\rm test}(t))$$

+ (Linear) **time-dependent** evolution equation

$$\frac{\partial \varphi_{\rm test}(t)}{\partial t} = \eta_{\rm bath}(t)\, \varphi_{\rm test}(t)$$

+ Good news
  - Noise is treated as **external**
  - Very constraining **spherical symmetry**

+ Motion solved using **Magnus series**

$$\varphi_{\rm test}(t) = \mathrm{e}^{\Omega(t)}\, \varphi_{\rm test}(0) \quad \text{with} \quad \Omega(t) = \int_0^t \mathrm{d}t'\, \eta_{\rm bath}(t')$$

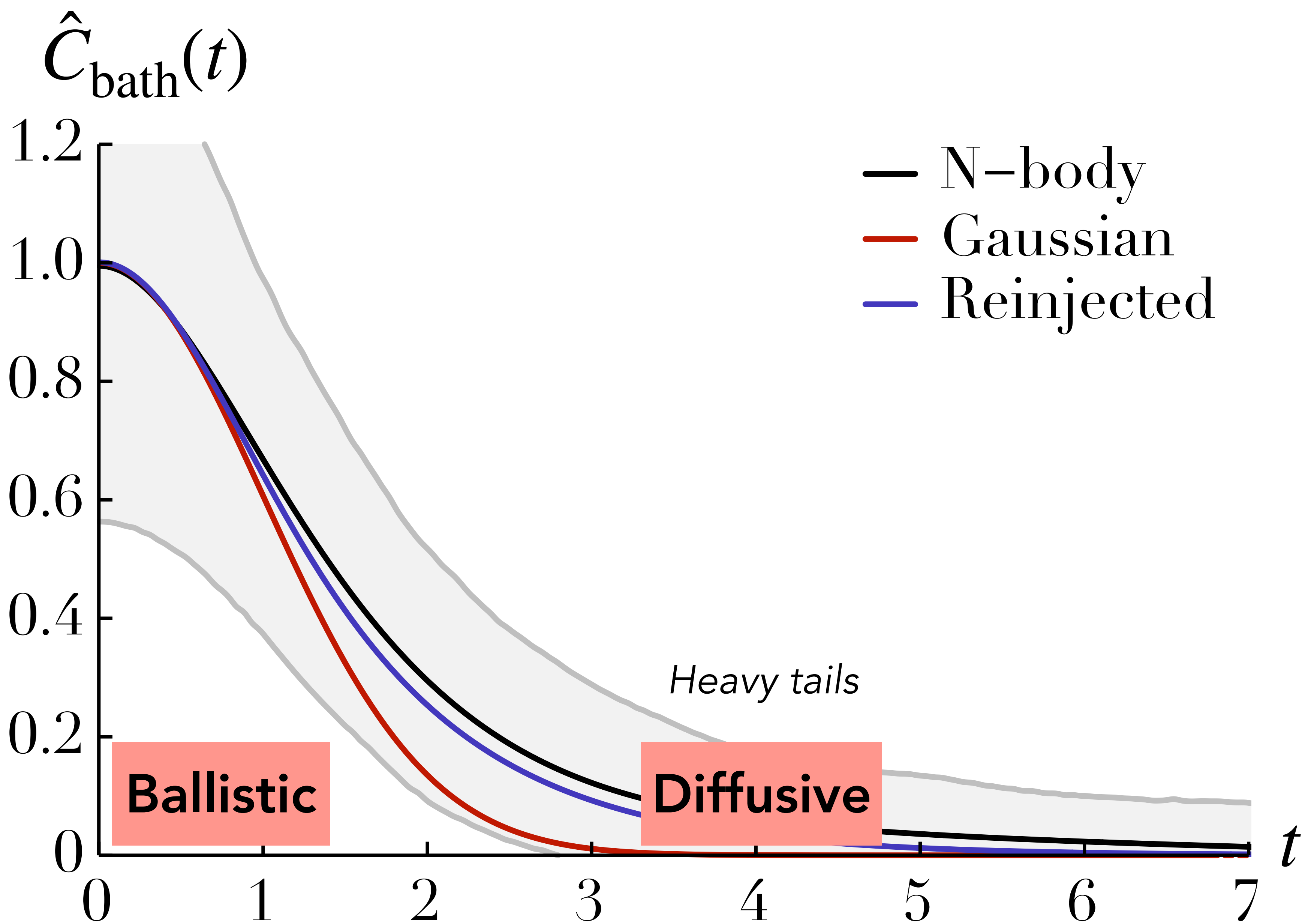
+ Explicit expression of the **time correlation**

$$\hat{C}_{\rm test}(t) = \exp\left[- \int_0^t \mathrm{d}t_1 \int_0^t \mathrm{d}t_2\, \hat{C}_{\rm bath}(t_1 - t_2) \right]$$

Improving the prediction

Imposing **self-consistency**

$$\frac{\mathrm{d}\hat{C}_{\mathrm{test}}}{\mathrm{d}t} = \frac{\mathrm{d}\hat{C}_{\mathrm{test}}}{\mathrm{d}t}\left[\hat{\mathrm{C}}_{\mathrm{bath}}\right] \quad \longleftrightarrow \quad \hat{C}_{\mathrm{bath}} = \left\langle \hat{C}_{\mathrm{test}} \right\rangle_{\mathrm{All\ particles}}$$



How to do better

Renormalisation

$$G^{-1} = G_0^{-1} - \Sigma$$

Asymptotic expansion

Additional relaxation

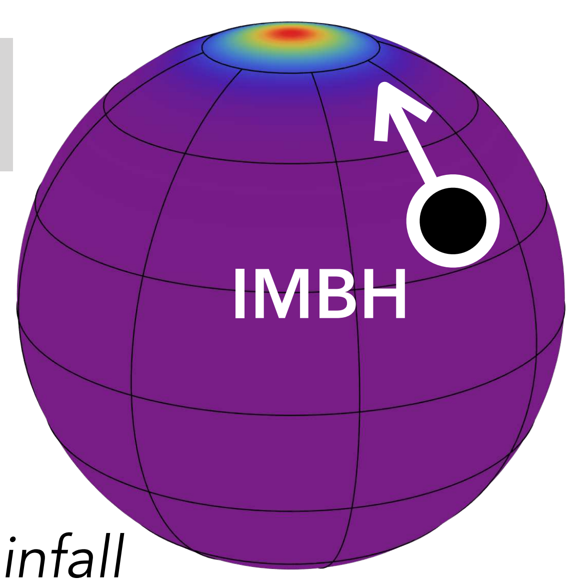
$$\frac{\mathrm{d}e}{\mathrm{d}t} \neq 0; \qquad \frac{\mathrm{d}a}{\mathrm{d}t} \neq 0$$

Impact of SRR and NR

Dynamical Friction

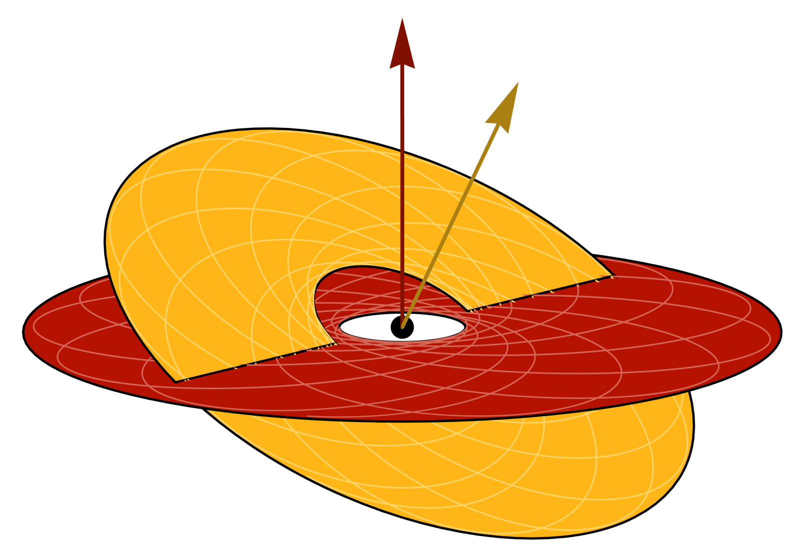
Ginat+(2023)

$\hat{\mathbf{L}}$



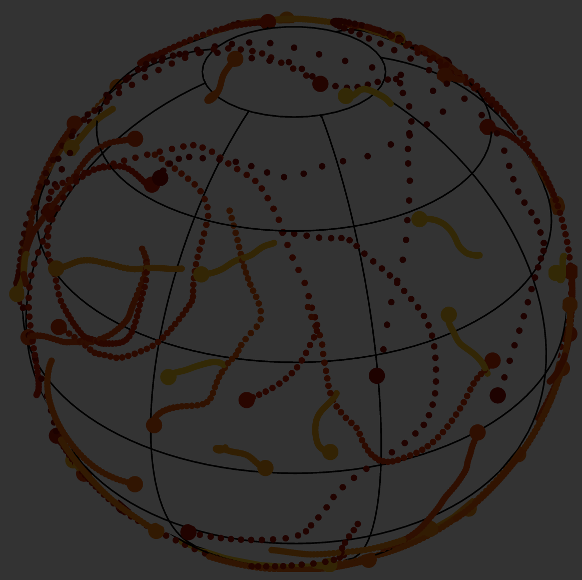
Orientation infall

Globular clusters

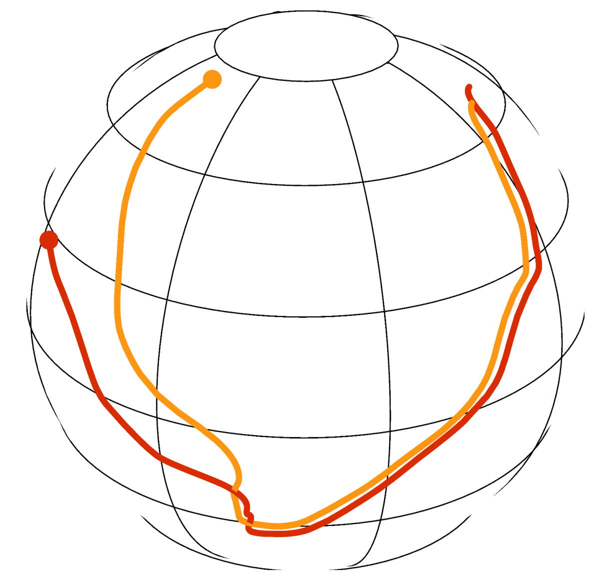


Some aspects of VRR

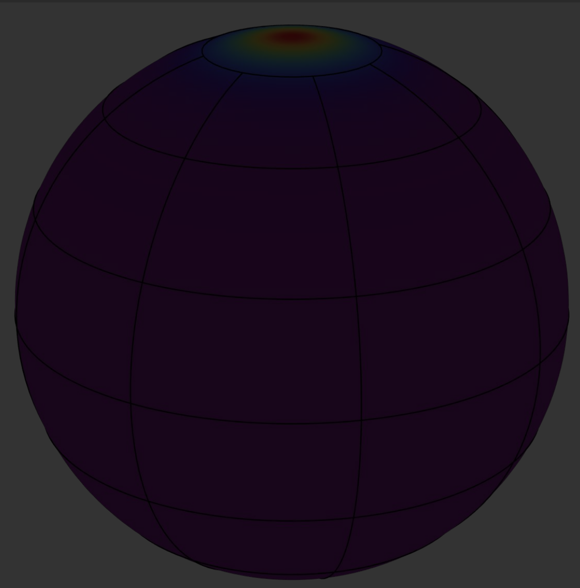
Statistics & Correlation



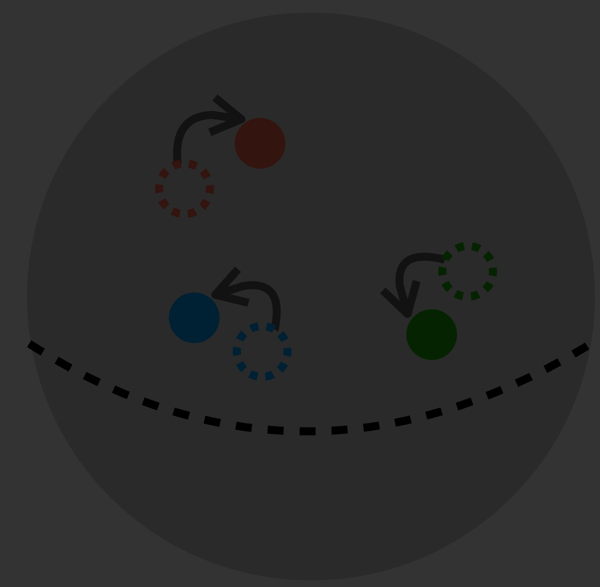
Diffusion & Dilution



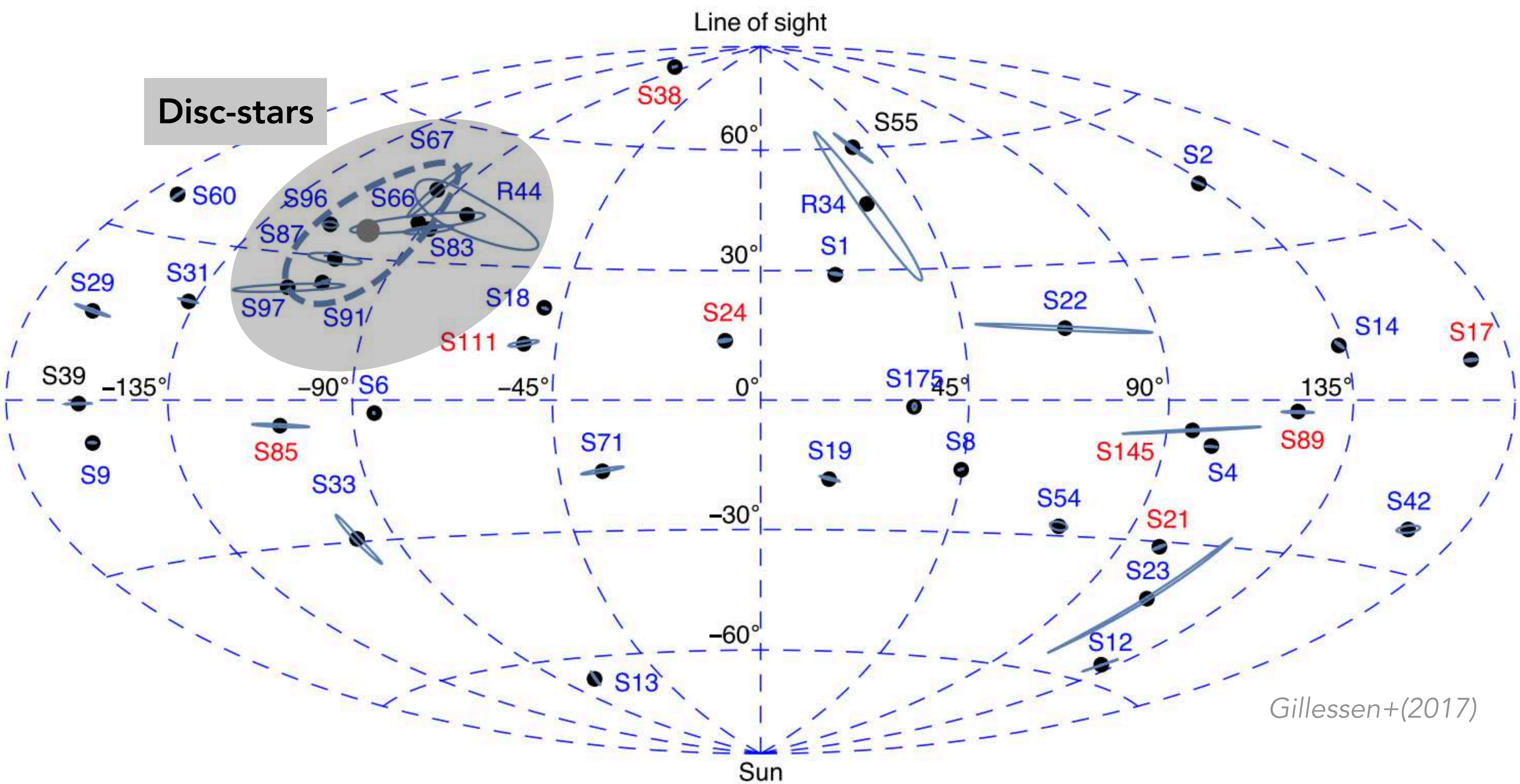
Thermodynamics



Numerical integration



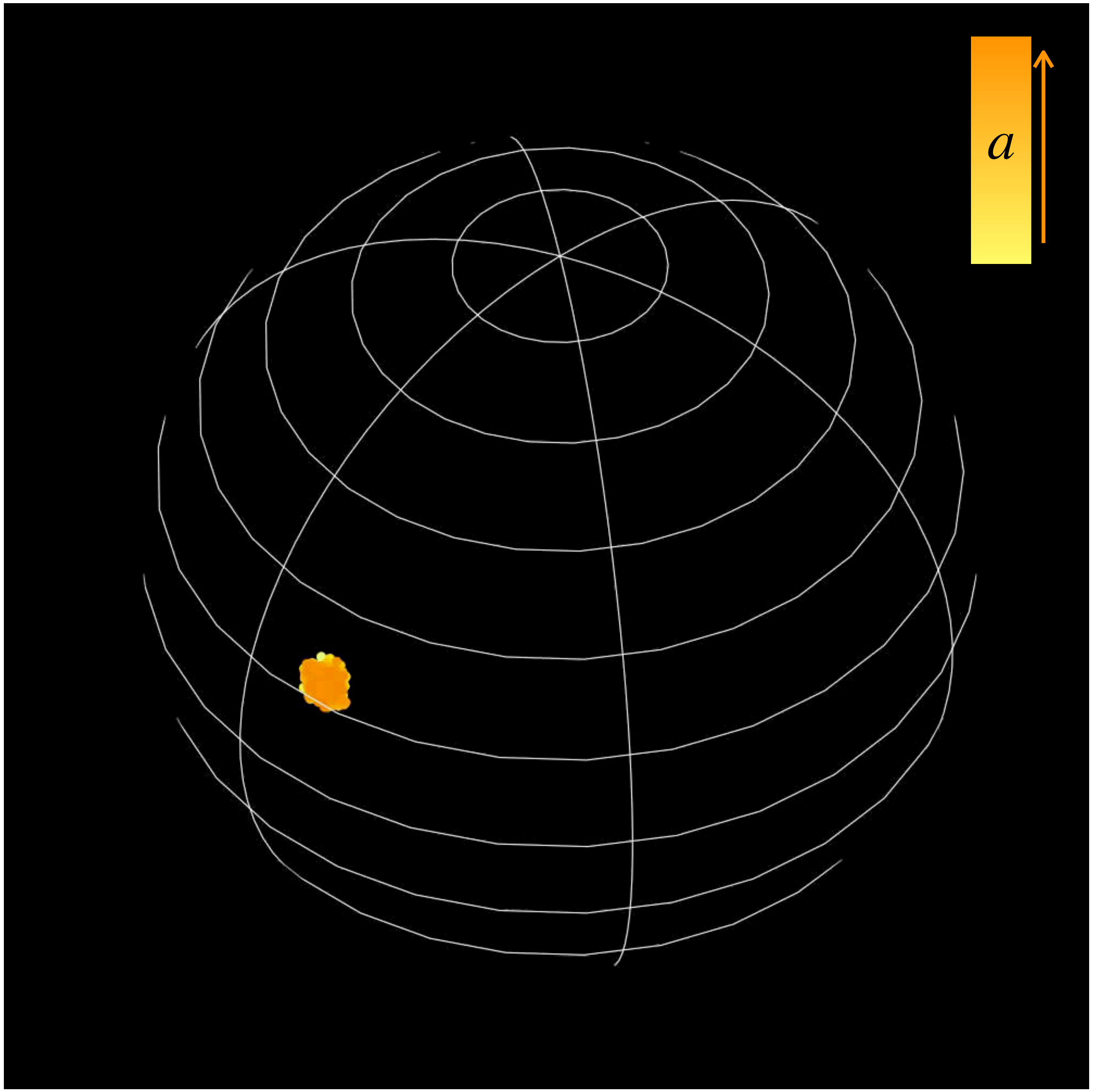
Vector Resonant Relaxation can affect the disc-stars



Gillessen+(2017)

How long should these stars stay ``**neighbors**''?

Vector Resonant Relaxation can randomize disc stars



+ How ``**neighbors**'' get separated

$$\frac{\mathrm{d}\hat{\mathbf{L}}_i}{\mathrm{d}t} = \eta(\hat{\mathbf{L}}_i, t)$$

+ Evolution sourced by a **shared**, **spatially-extended** and **time-correlated** noise

$$\left\langle \eta(a_i, \hat{\mathbf{L}}_i, t)\, \eta(a_j, \hat{\mathbf{L}}_j, t') \right\rangle = C(a_i, a_j, \hat{\mathbf{L}}_i \cdot \hat{\mathbf{L}}_j, t - t')$$

+ Two joint sources of **separation**

- **Parametric** separation

$$a_i \neq a_j$$

- **Angular** separation

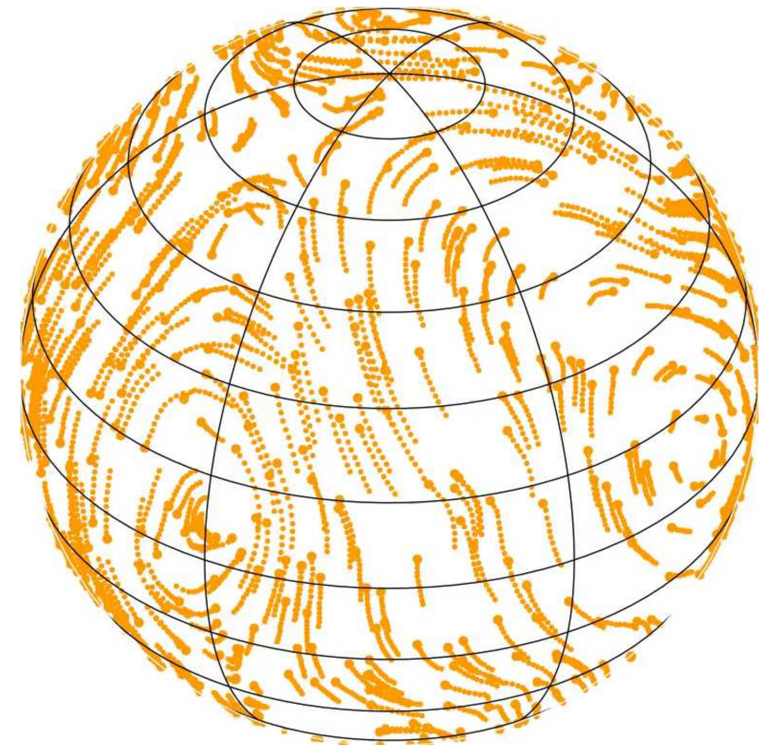
$$\hat{\mathbf{L}}_i \neq \hat{\mathbf{L}}_j$$

VRR around SgrA*

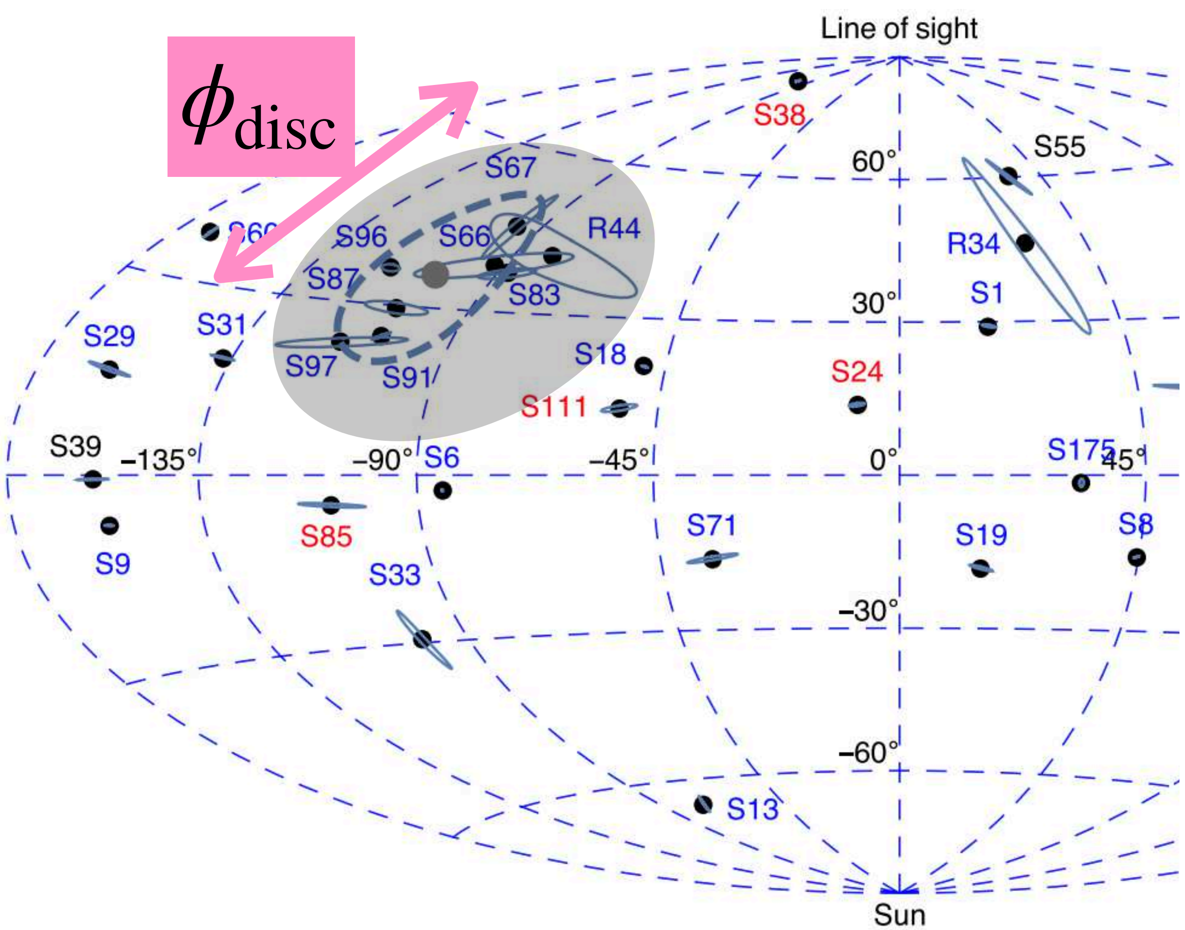
Model

- Old stars
(*unresolved but relaxed)*
- **IMBHs**
(strong source of Poisson noise)
- S-stars disc ICs
(initial angular dispersion)

Kinetic theory

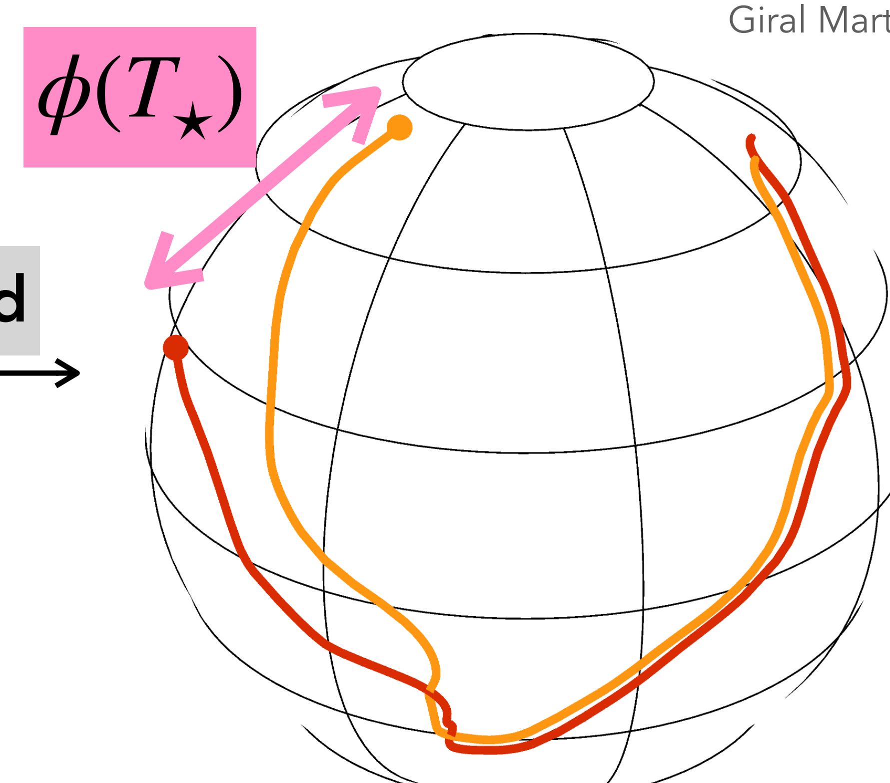


Dilution



Likelihood

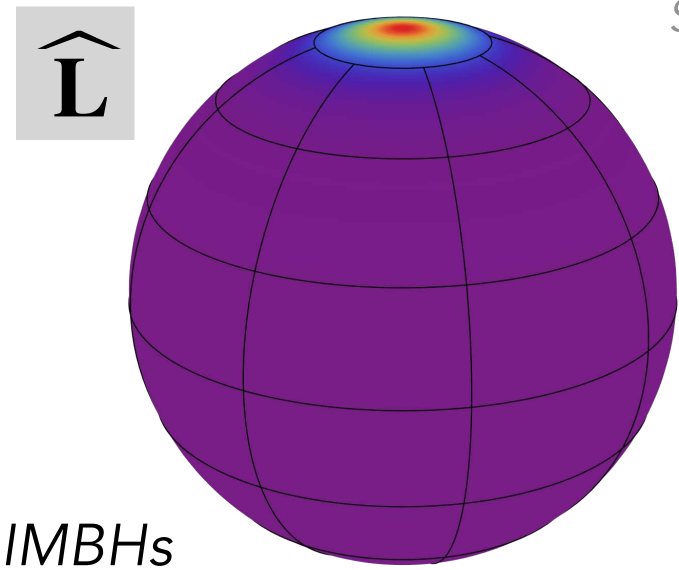
Giral Martinez+(2020)



How to do better

Anisotropic orientations

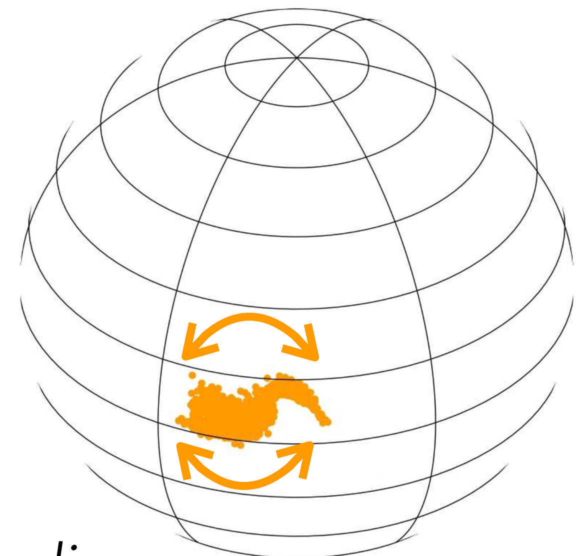
Szolgyen+(2018)



Disc of IMBHs

Self-gravity

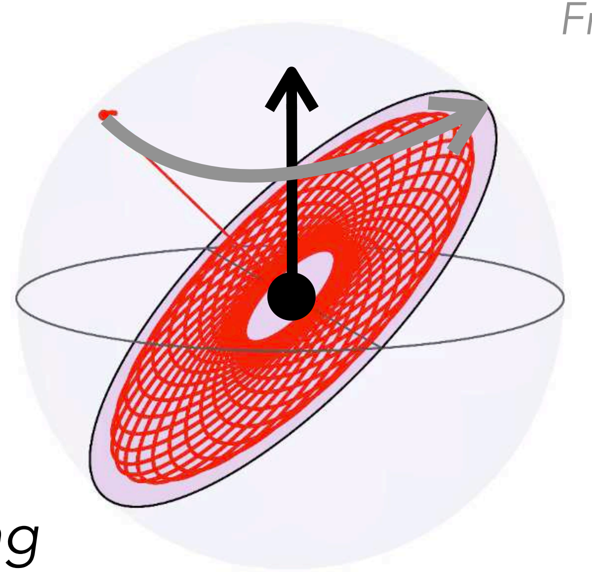
Kocsis+(2011)



Pairwise couplings

Lense-Thirring

Fragione+(2022)



Phase mixing

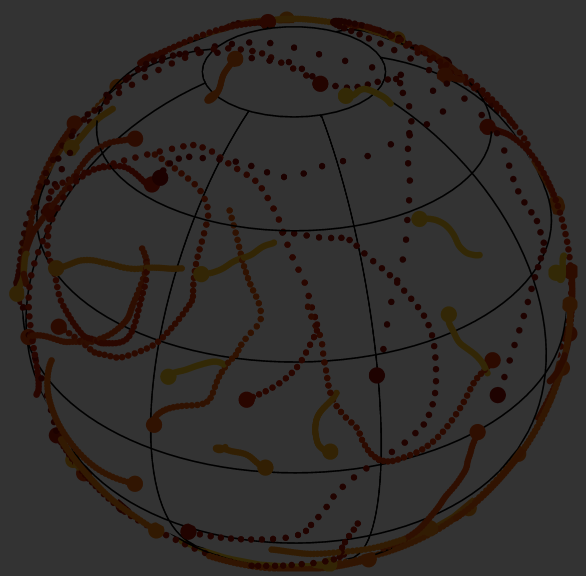
Faking the dilution

$$\frac{\mathrm{d}(\hat{\mathbf{L}}_1 \cdot \hat{\mathbf{L}}_2)}{\mathrm{d}t} = \eta[\hat{\mathbf{L}}_1, \hat{\mathbf{L}}_2, t]$$

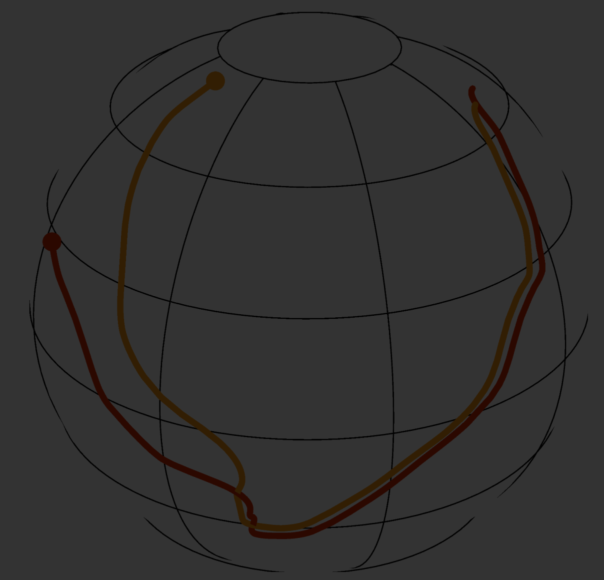
Langevin equation

Some aspects of VRR

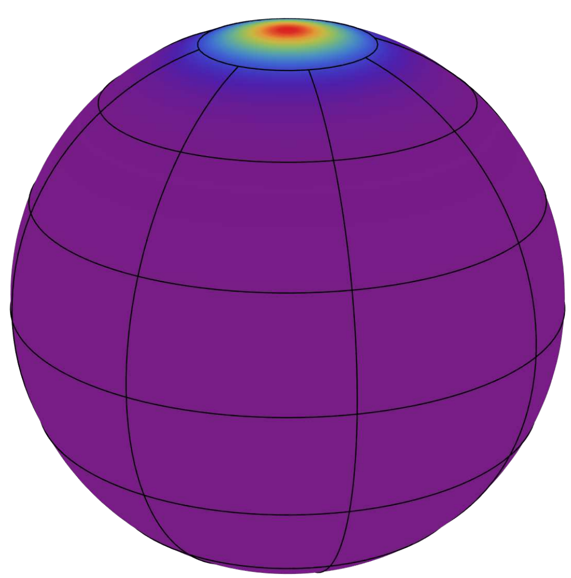
Statistics & Correlation



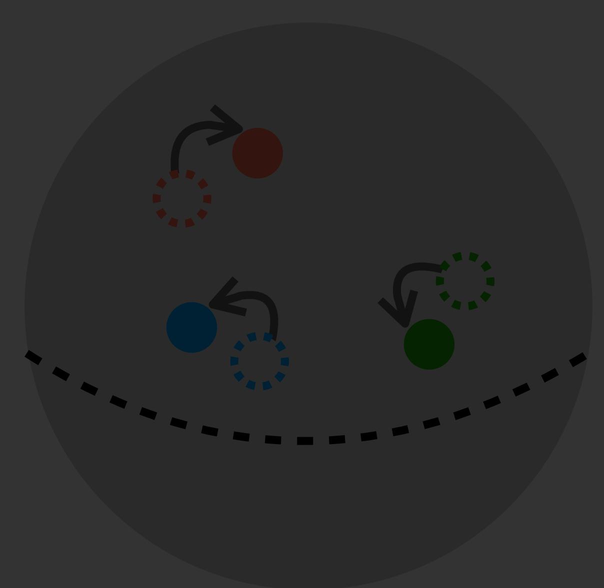
Diffusion & Dilution



Thermodynamics



Numerical integration



Thermodynamics of VRR

N-body dynamics

$$\frac{\partial F_{\mathrm{d}}}{\partial t} + \left[F_{\mathrm{d}}, H(F_{\mathrm{d}})\right] = 0$$

Quadratic, orbit-averaged, hierarchical, multi-population

Thermodynamics of VRR

N-body dynamics

$$\frac{\partial F_{\mathrm{d}}}{\partial t} + \left[F_{\mathrm{d}}, H(F_{\mathrm{d}})\right] = 0$$

Quadratic, orbit-averaged,
hierarchical, multi-population

Kinetic Theory

$$\frac{\partial \langle F_{\mathrm{d}} \rangle}{\partial t} = C\left[\langle F_{\mathrm{d}} \rangle, \langle F_{\mathrm{d}} \rangle\right]$$

Integrable equilibrium,
small perturbations, quasi-linear expansion,
collective effects, resonant couplings

Thermodynamics of VRR

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Thermodynamics

$$F_{\mathrm{eq}}(\hat{\mathbf{L}}) = \lim_{t \to +\infty} \langle F_{\mathrm{d}}(\hat{\mathbf{L}}, t) \rangle$$

Ergodic principle

Thermodynamics of VRR

N-body dynamics

$$\frac{\partial F_{\mathrm{d}}}{\partial t} + \left[F_{\mathrm{d}}, H(F_{\mathrm{d}})\right] = 0$$

Quadratic, orbit-averaged, hierarchical, multi-population

Kinetic Theory

$$\frac{\partial \langle F_{\mathrm{d}} \rangle}{\partial t} = C\left[\langle F_{\mathrm{d}} \rangle, \langle F_{\mathrm{d}} \rangle\right]$$

Integrable equilibrium, small perturbations, quasi-linear expansion, collective effects, resonant couplings

Thermodynamics

$$F_{\mathrm{eq}}(\widehat{\mathbf{L}}) = \lim_{t \to +\infty} \langle F_{\mathrm{d}}(\widehat{\mathbf{L}}, t) \rangle$$

Ergodic principle

Global N-body invariants

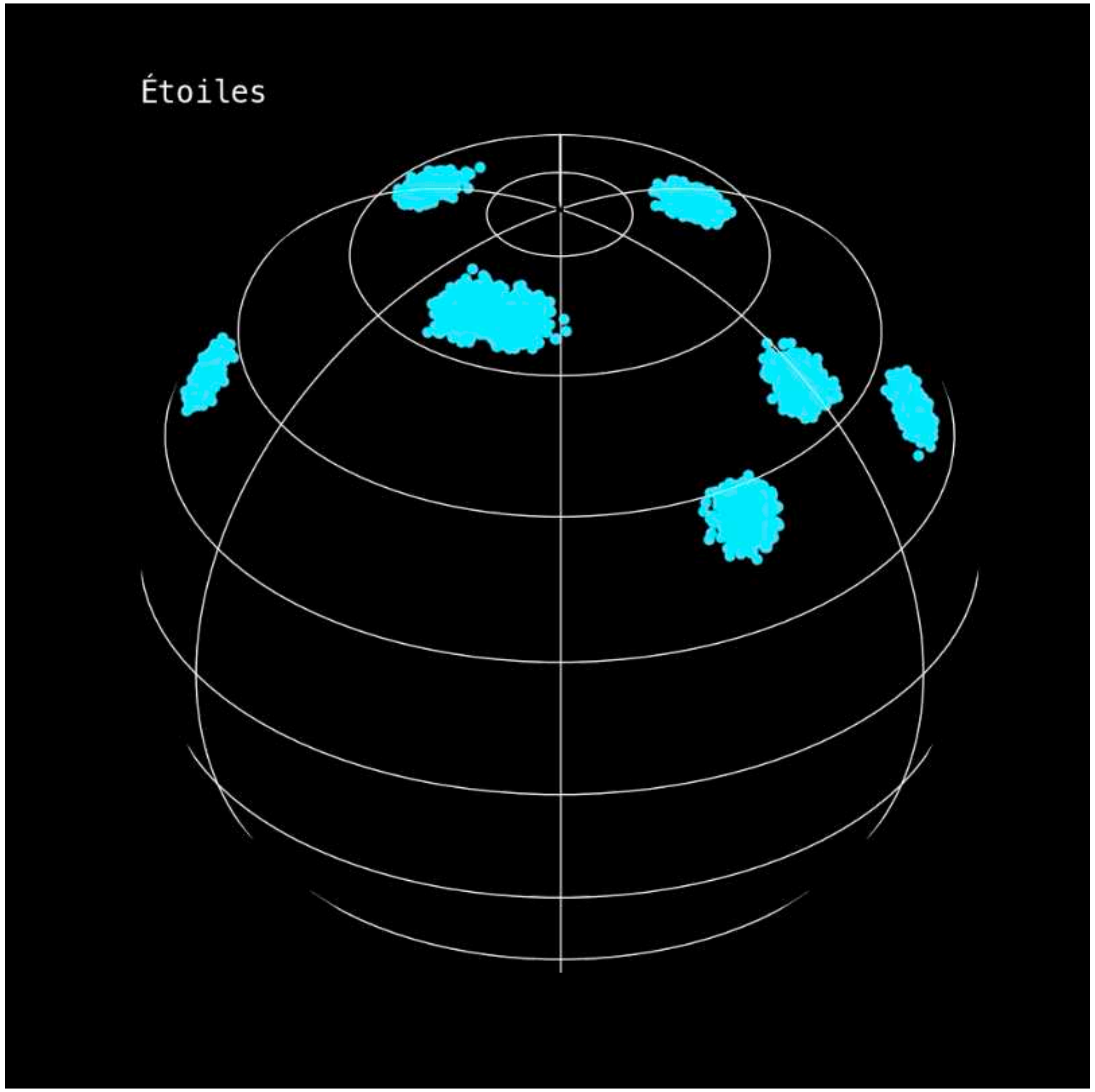
$$\mathbf{K} = (m, a, e)$$

Annuli shape

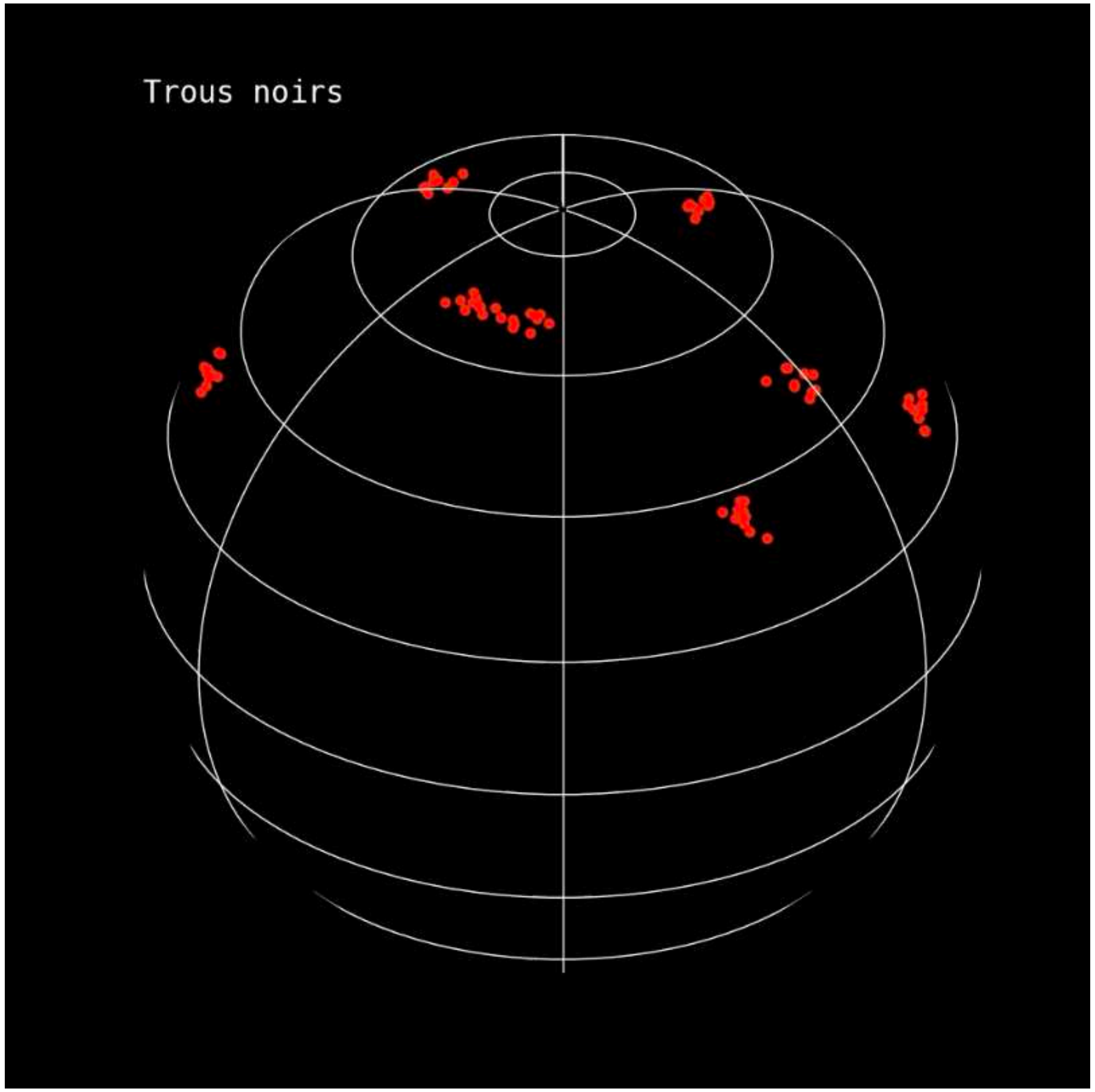
$$\begin{cases} N(\mathbf{K}) & \textit{Sub-populations} \\ E_{\mathrm{tot}} & \textit{Total energy} \\ \widehat{\mathbf{L}}(\mathbf{K}) & \textit{Total angular momentum} \end{cases}$$

An example of relaxation

Stars



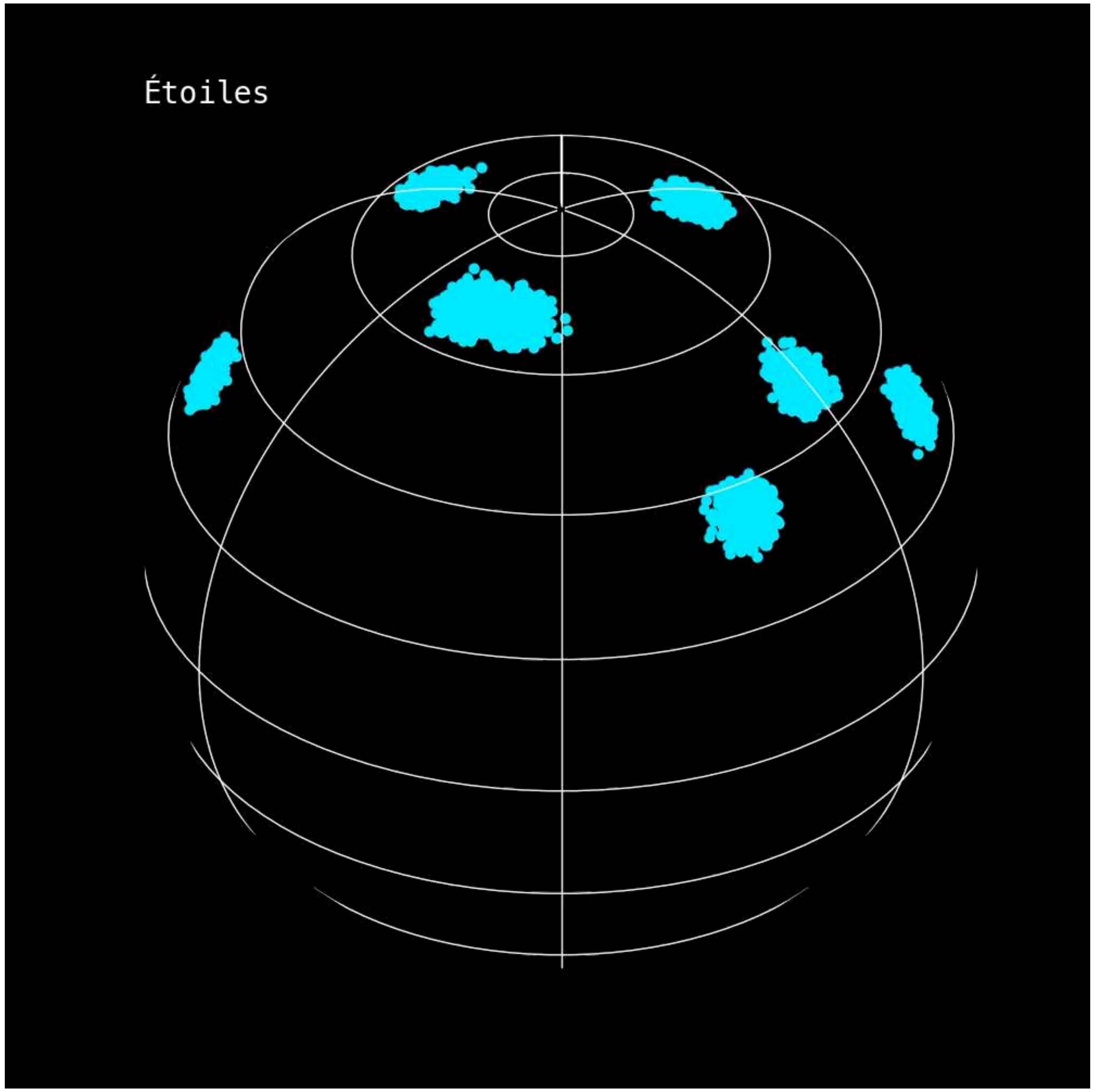
Intermediate mass black holes



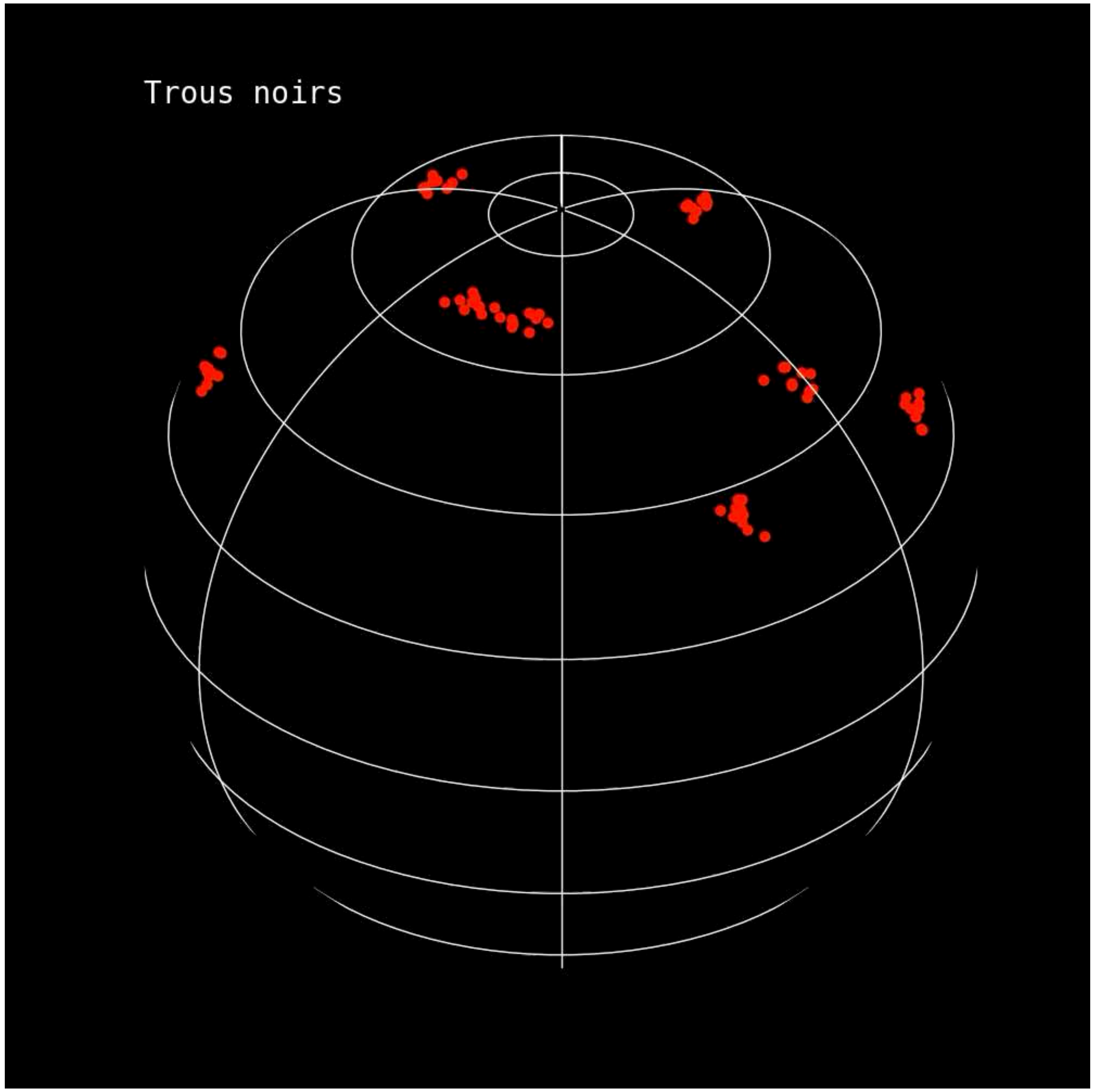
Anisotropic distribution from gas infall

An example of relaxation

Stars



Intermediate mass black holes

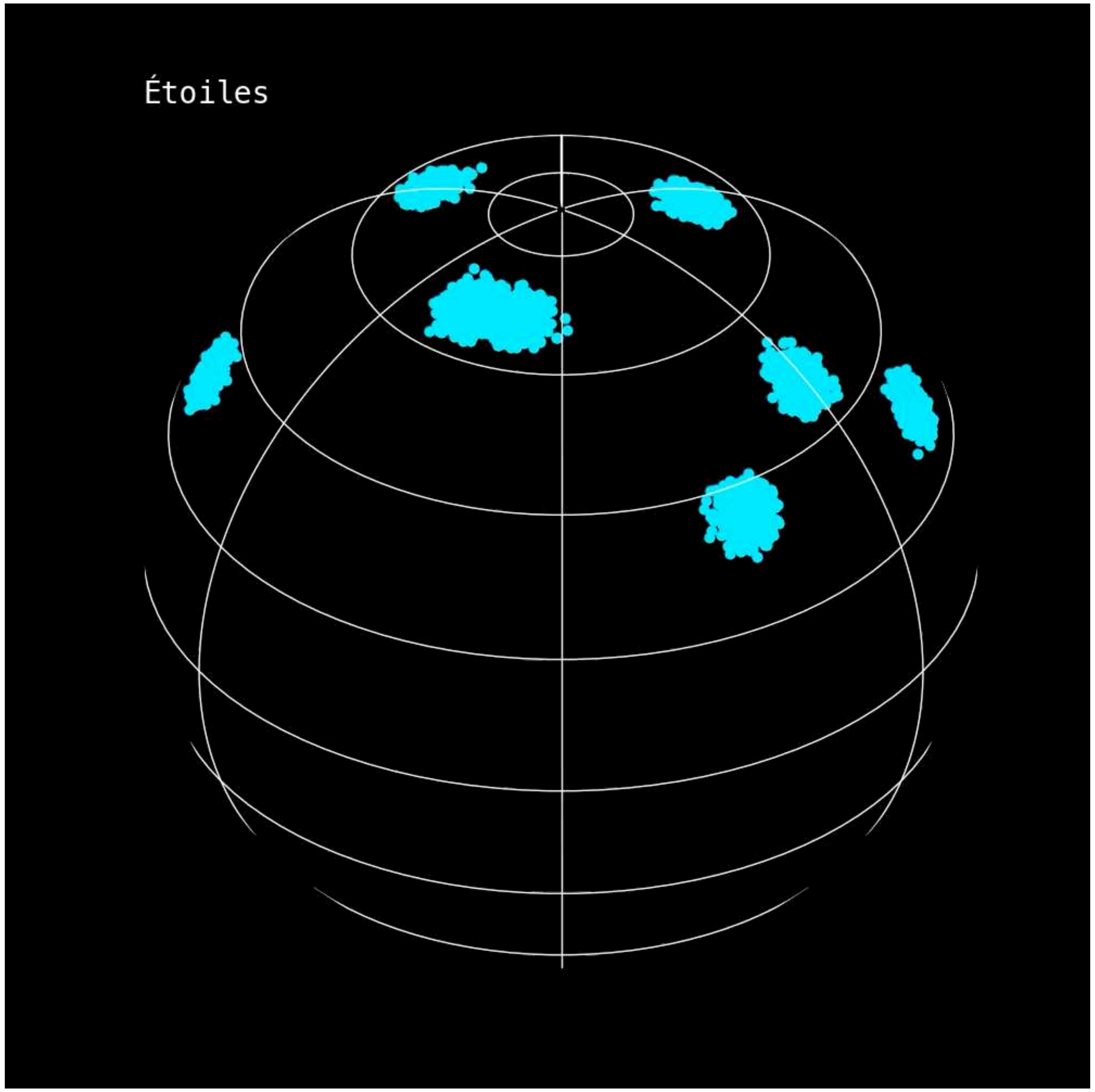


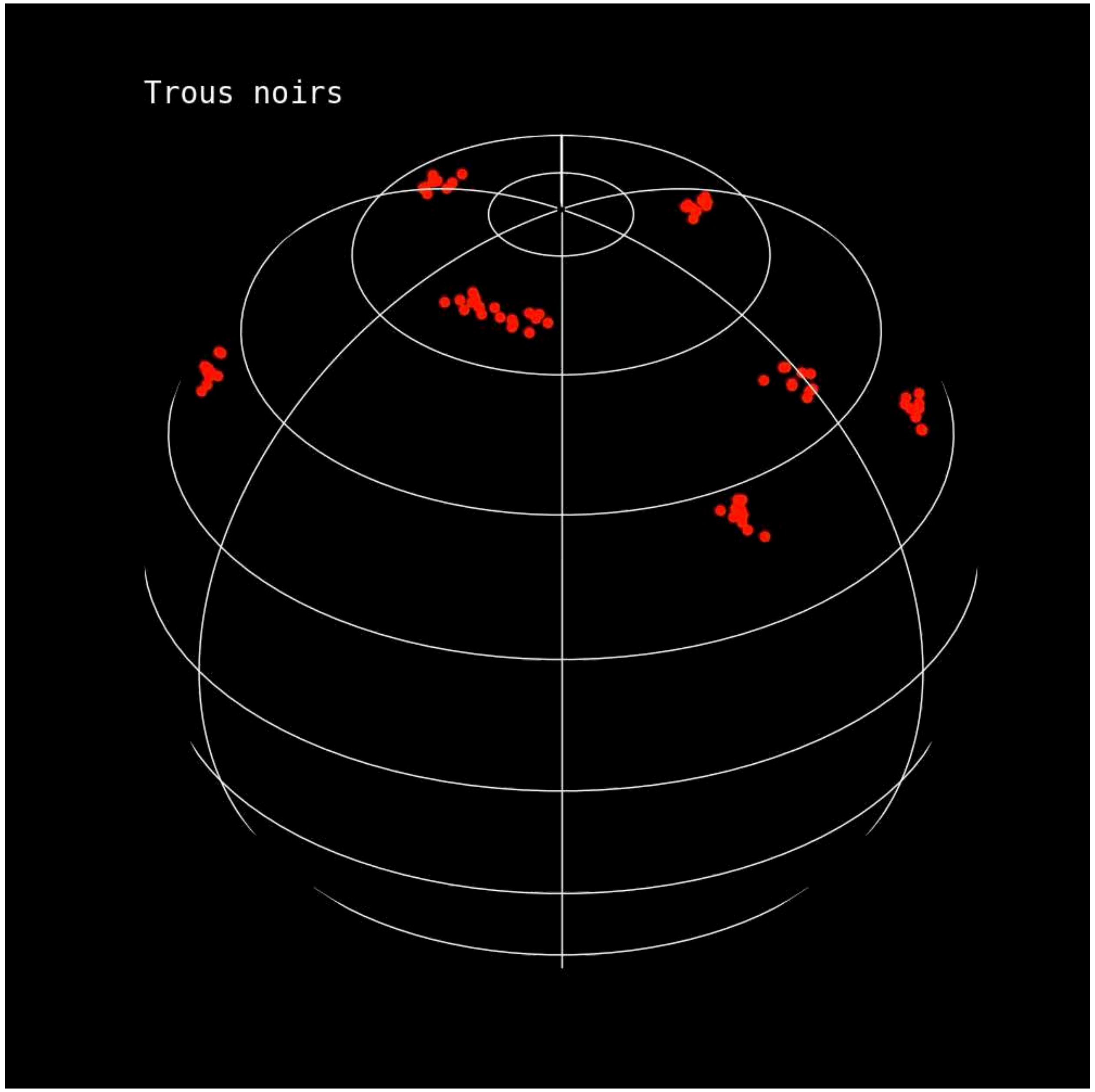
Relaxation on **short timescales**

An example of relaxation

Stars

Intermediate mass black holes



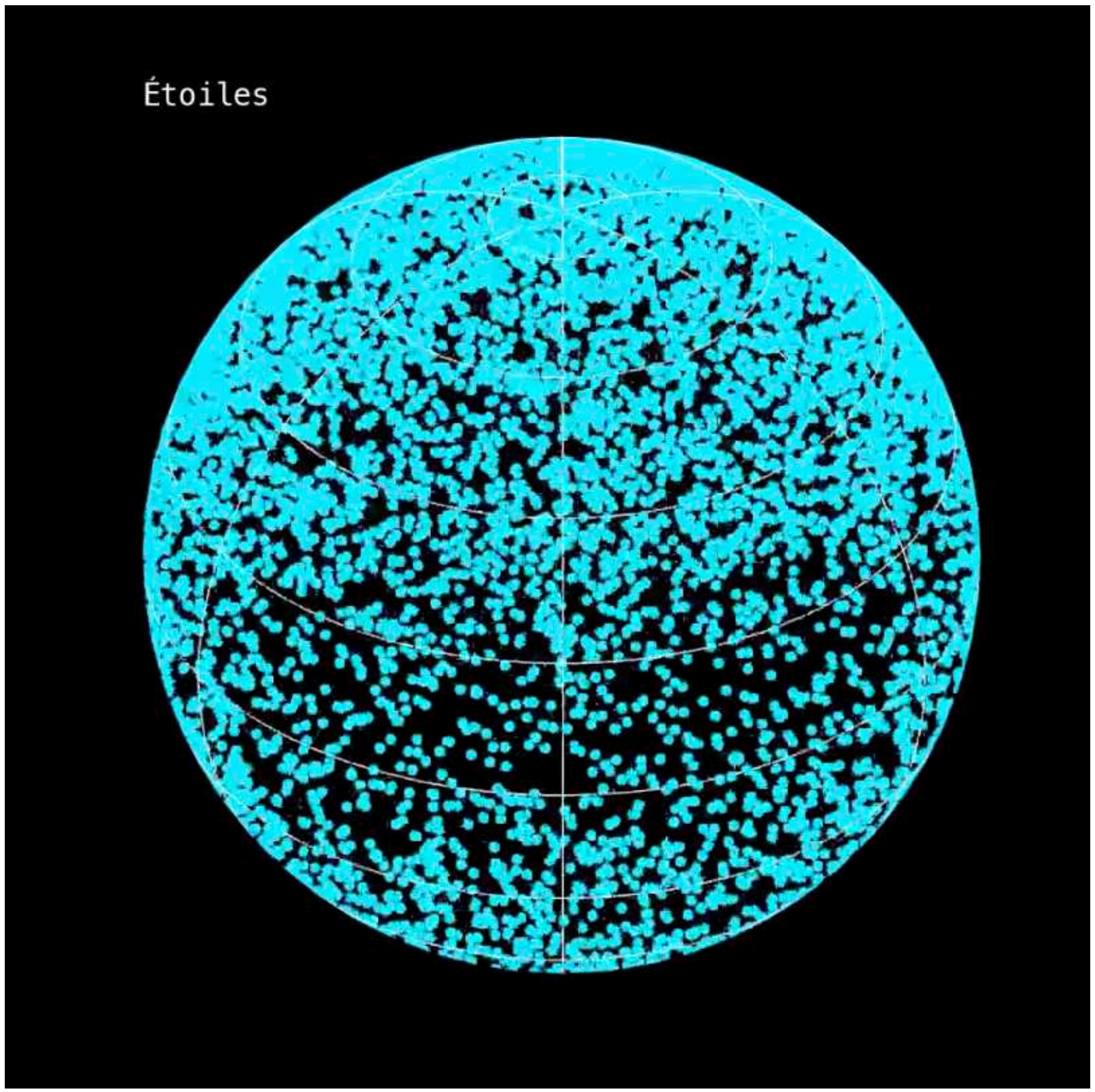


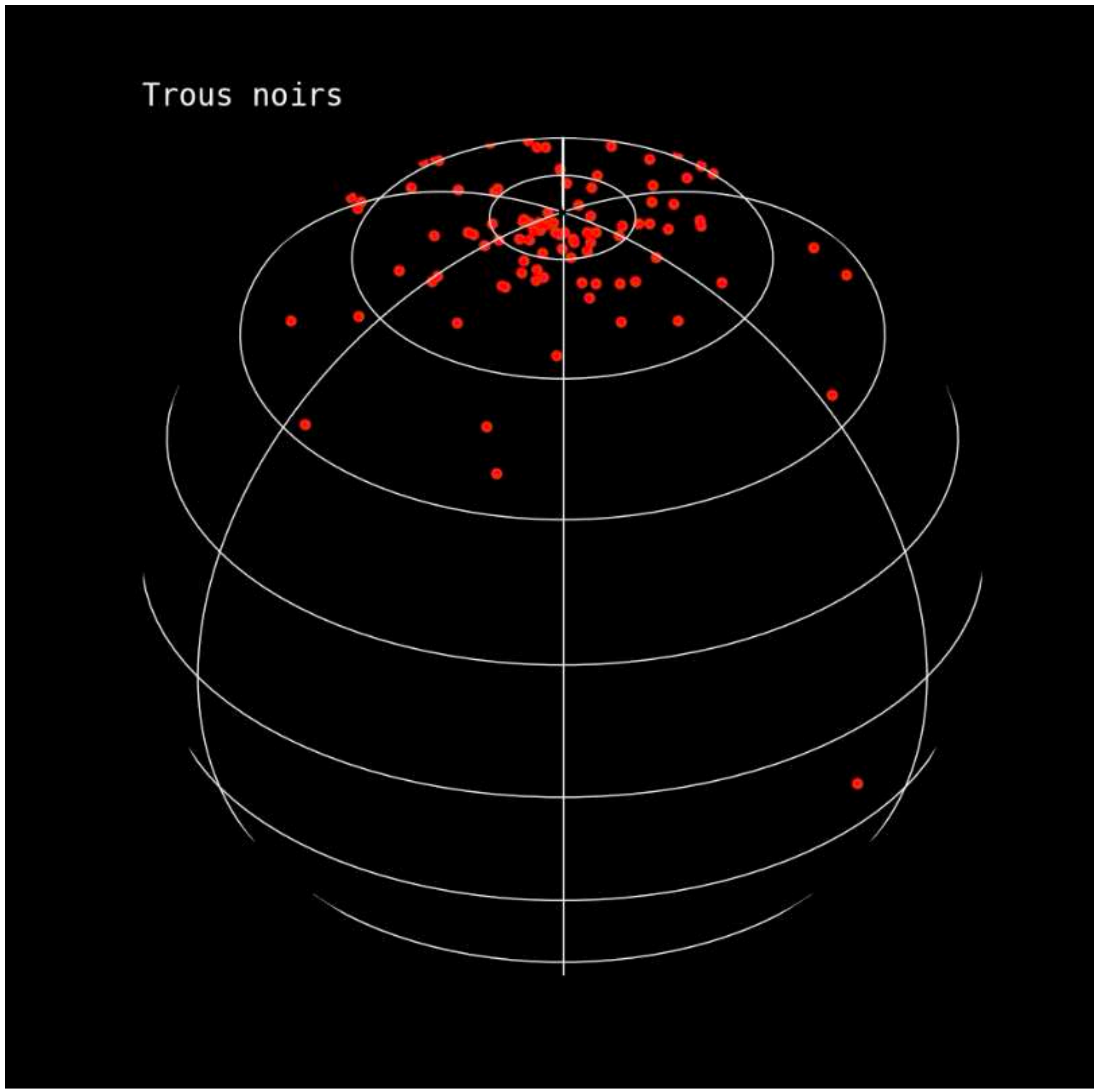
Relaxation on **long timescales**

An example of relaxation

Stars

Intermediate mass black holes





Anisotropic equilibria

Thermodynamics of VRR

Entropy maximisation

$$S \propto \int \mathrm{d}\hat{\mathbf{L}}\mathrm{d}\mathbf{K}\, F \ln[F]$$

under the conservation of the **invariants**

Generalised **Boltzmann DF**

$$F_{\mathrm{eq}}(\hat{\mathbf{L}}, \mathbf{K}) \propto \exp\Big[-\beta\, \varepsilon(\hat{\mathbf{L}}, \mathbf{K}) + L(\mathbf{K})\, \boldsymbol{\gamma} \cdot \hat{\mathbf{L}} \Big]$$

Temperature *Spin*

Self-consistency

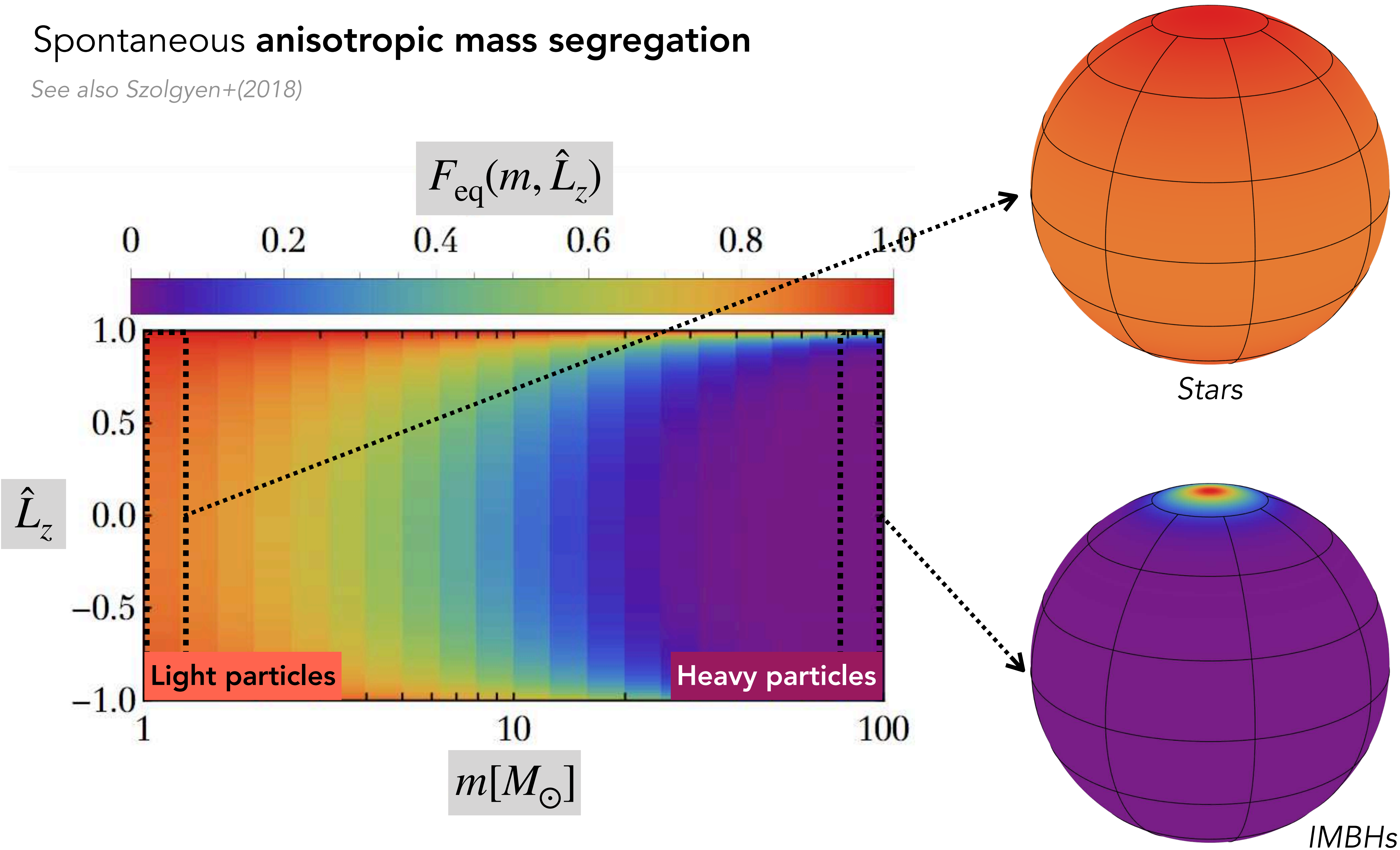
$$\big[\beta, \boldsymbol{\gamma}\big] \longrightarrow \big[E_{\mathrm{tot}}, \mathbf{L}_{\mathrm{tot}}\big] \stackrel{?}{=} \big[E_{\mathrm{tot}}(t=0), \mathbf{L}_{\mathrm{tot}}(t=0)\big]$$

Phase transition between ordered and disordered states

An example of equilibrium

Spontaneous **anisotropic mass segregation**

See also Szolgyen+(2018)



The more individually massive the population, the stronger the **alignment**

How to do better

Non-axisymmetry

$$\langle Y_{\ell m} \rangle \quad \text{for} \quad m \neq 0$$

Spontaneous symmetry breaking

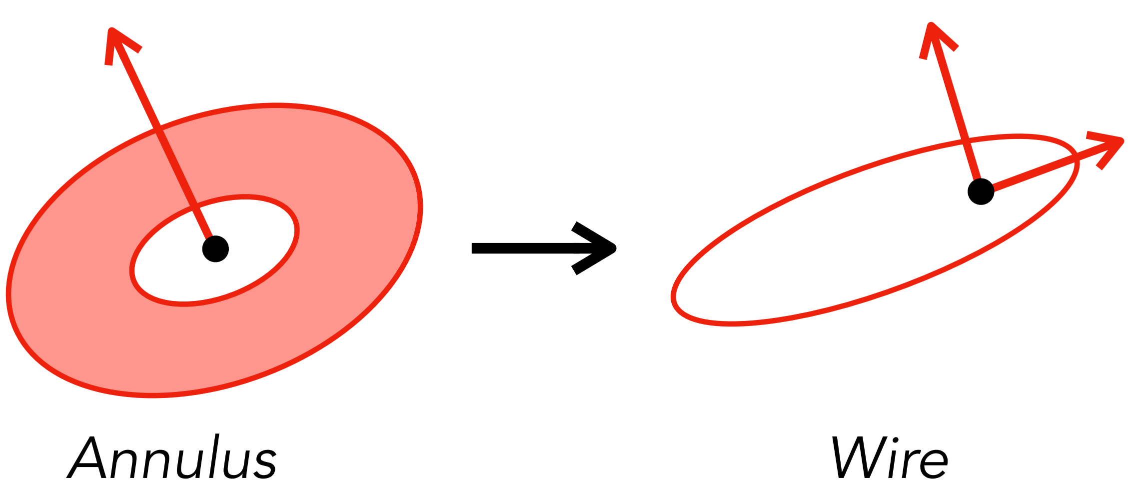
Timescale

$$F_{\mathrm{b}}(\hat{\mathbf{L}}, t) \xrightarrow{T_{\mathrm{relax}}} F_{\mathrm{eq}}(\hat{\mathbf{L}})$$

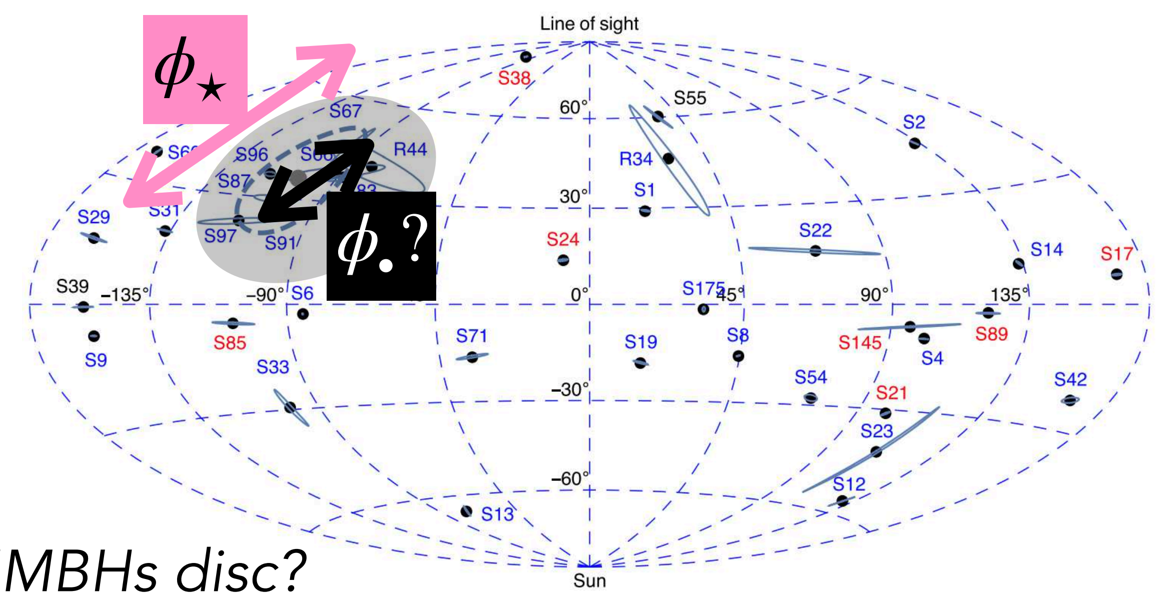
How fast to create anisotropies?

Wires thermodynamics

Gruzinov+(2020)



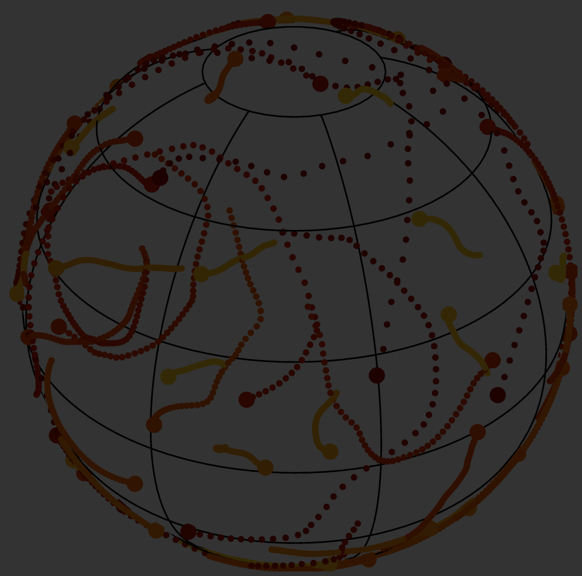
Observations



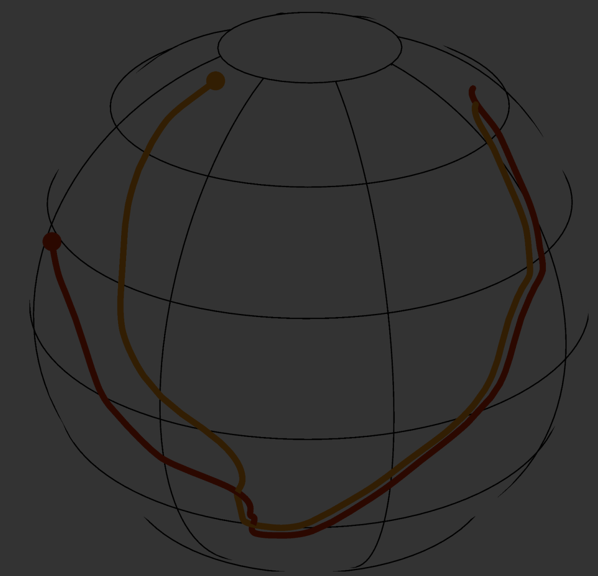
IMBHs disc?

Some aspects of VRR

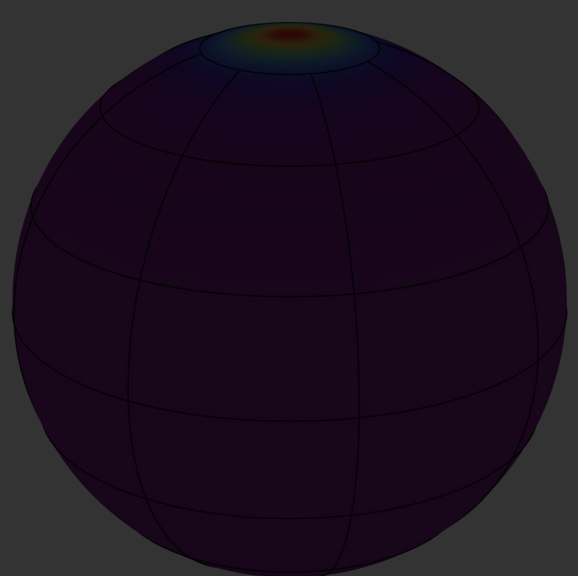
Statistics & Correlation



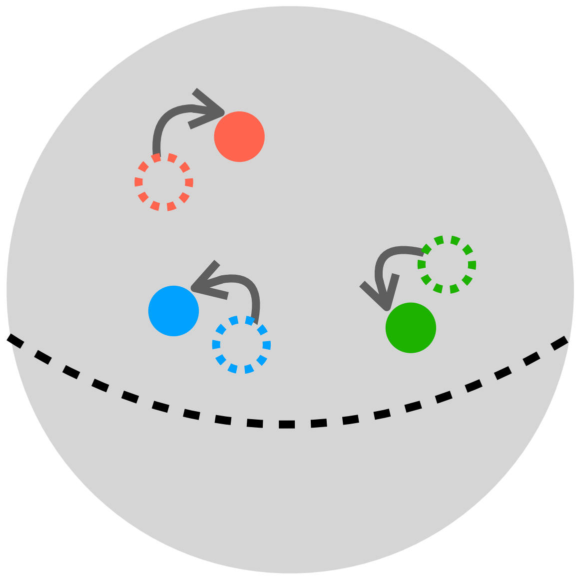
Diffusion & Dilution



Thermodynamics



Numerical integration



Time integration

Classical integration

$$\dot{y} = F(y)$$

$$y_n \xrightarrow{\tau} y_{n+1}$$

Time integration

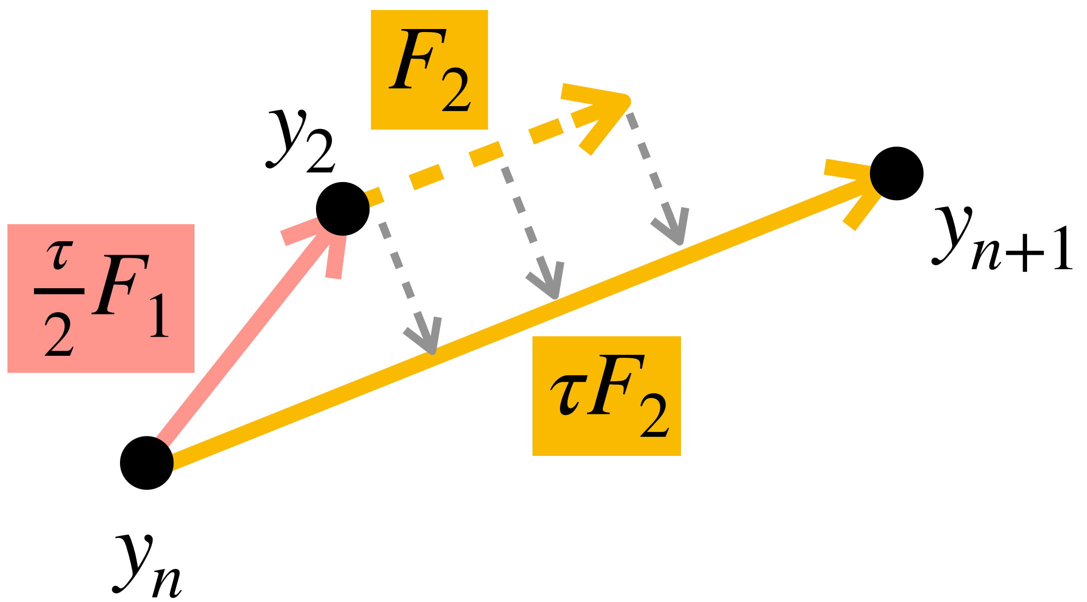
Classical integration

$$\dot{y} = F(y)$$

$$y_n \xrightarrow{\tau} y_{n+1}$$

Explicit Midpoint rule

$$\begin{aligned} F_1 &= F(y_n) \\ y_2 &= y_n + \frac{1}{2}\,\tau\,F_1 \\ F_2 &= F(y_2) \\ y_{n+1} &= y_n + \tau\,F_2 \end{aligned}$$



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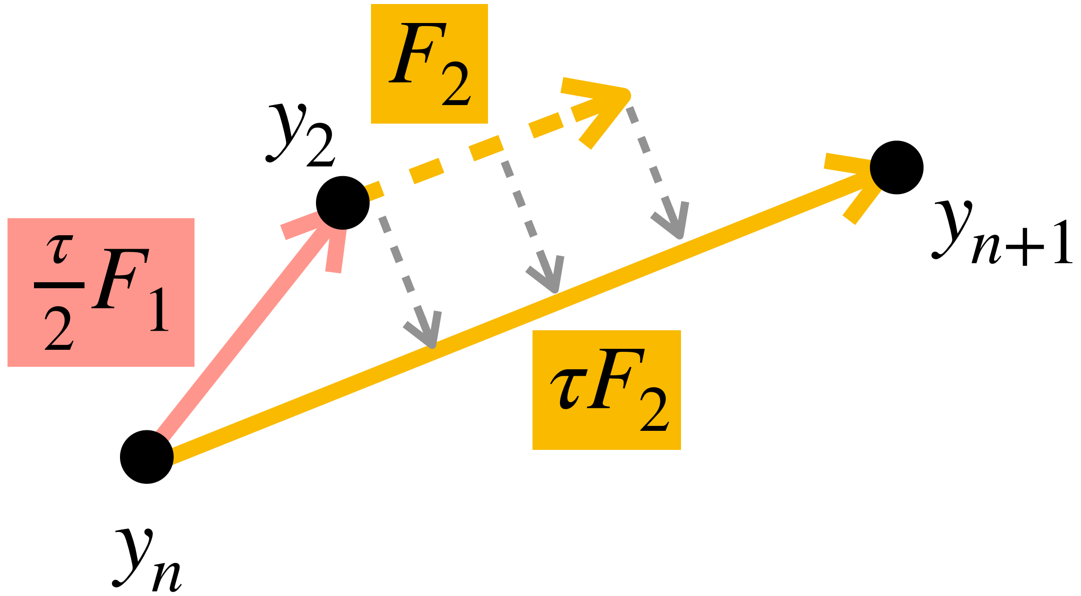


Fourth-order **Runge-Kutta**

$$\begin{aligned} F_1 &= F(y_n) \\ y_2 &= y_n + \frac{1}{2}\tau F_1 \\ F_2 &= F(y_2) \\ y_3 &= y_n + \frac{1}{2}\tau F_2 \\ F_3 &= F(y_3) \\ y_4 &= y_n + \tau F_3 \\ F_4 &= F(y_4) \\ F &= \frac{1}{6}F_1 + \frac{1}{3}F_2 + \frac{1}{3}F_3 + \frac{1}{6}F_4 \\ y_{n+1} &= y_n + \tau F \end{aligned}$$

Time integration

Classical integration

$$\dot{y} = F(y)$$

$$y_n \xrightarrow{\tau} y_{n+1}$$

Explicit Midpoint rule

$$F_1 = F(y_n)$$
$$y_2 = y_n + \frac{1}{2}\tau F_1$$
$$F_2 = F(y_2)$$
$$y_{n+1} = y_n + \tau F_2$$

How to comply with constraints?

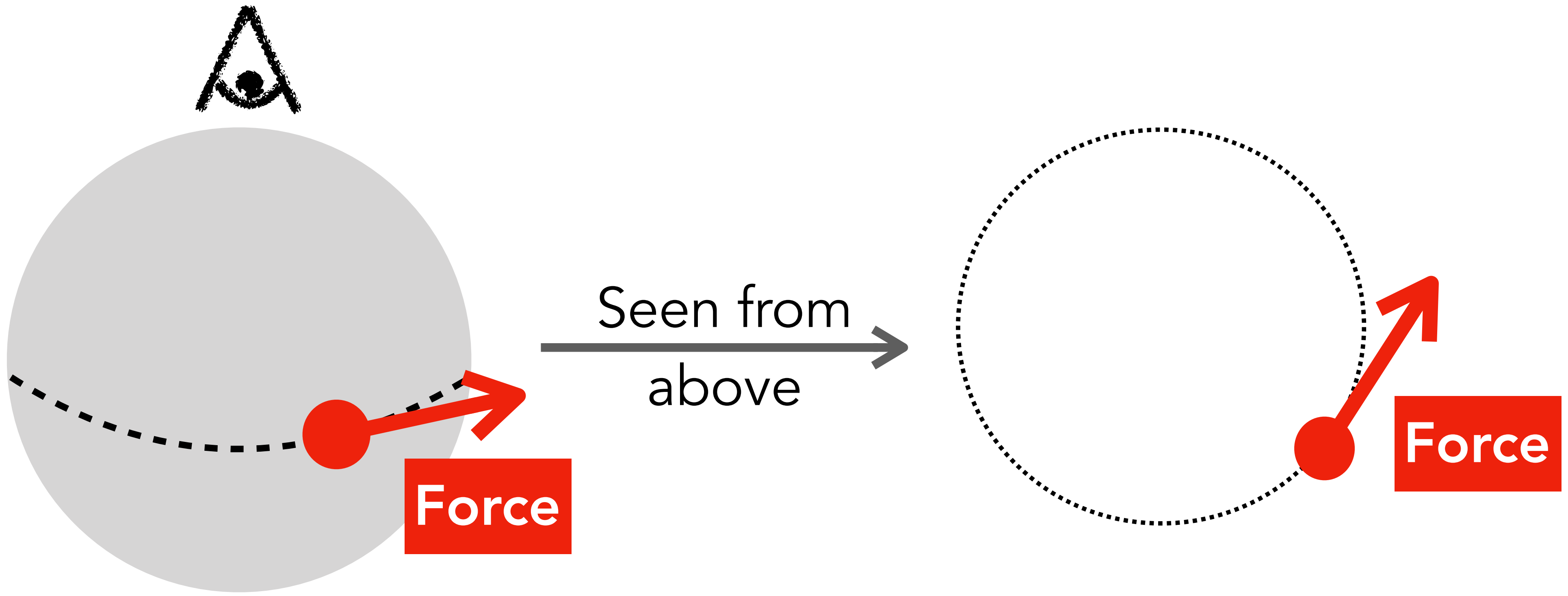
$$y' = y + \tau F$$
$$F = F_1 + F_2$$

Fourth-order **Runge-Kutta**

$$F_1 = F(y_n)$$
$$y_2 = y_n + \frac{1}{2}\tau F_1$$
$$F_2 = F(y_2)$$
$$y_3 = y_n + \frac{1}{2}\tau F_2$$
$$F_3 = F(y_3)$$
$$y_4 = y_n + \tau F_3$$
$$F_4 = F(y_4)$$
$$F = \frac{1}{6}F_1 + \frac{1}{3}F_2 + \frac{1}{3}F_3 + \frac{1}{6}F_4$$
$$y_{n+1} = y_n + \tau F$$

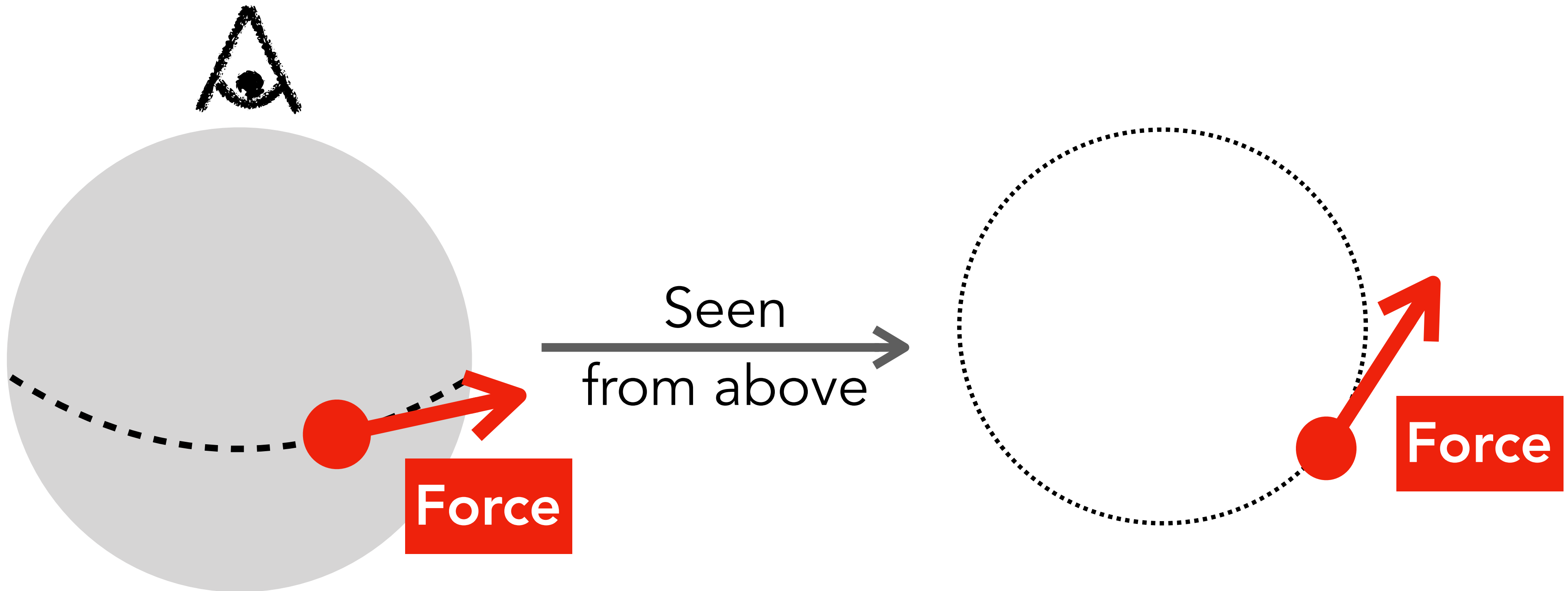
Time integration

Force on a star

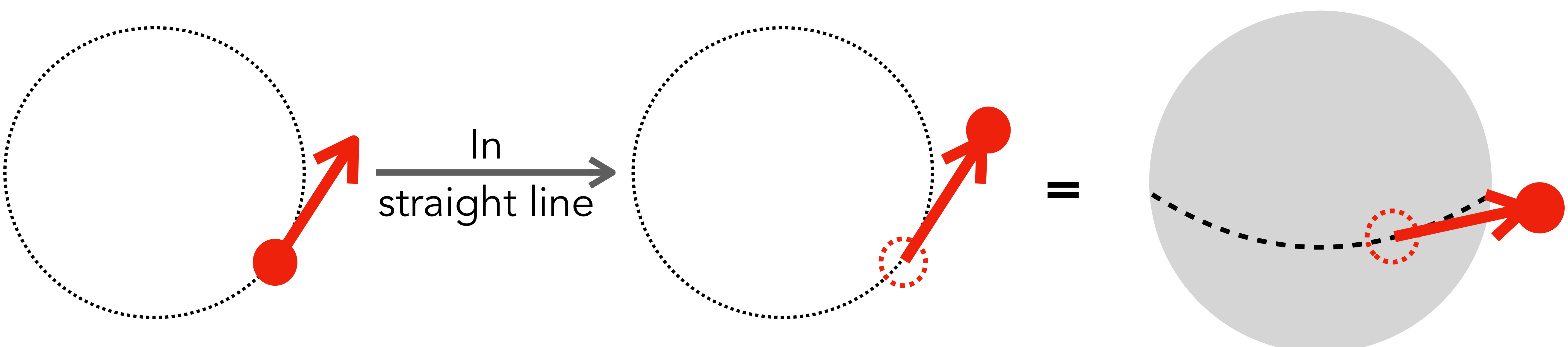


Classical integration

Force on a star



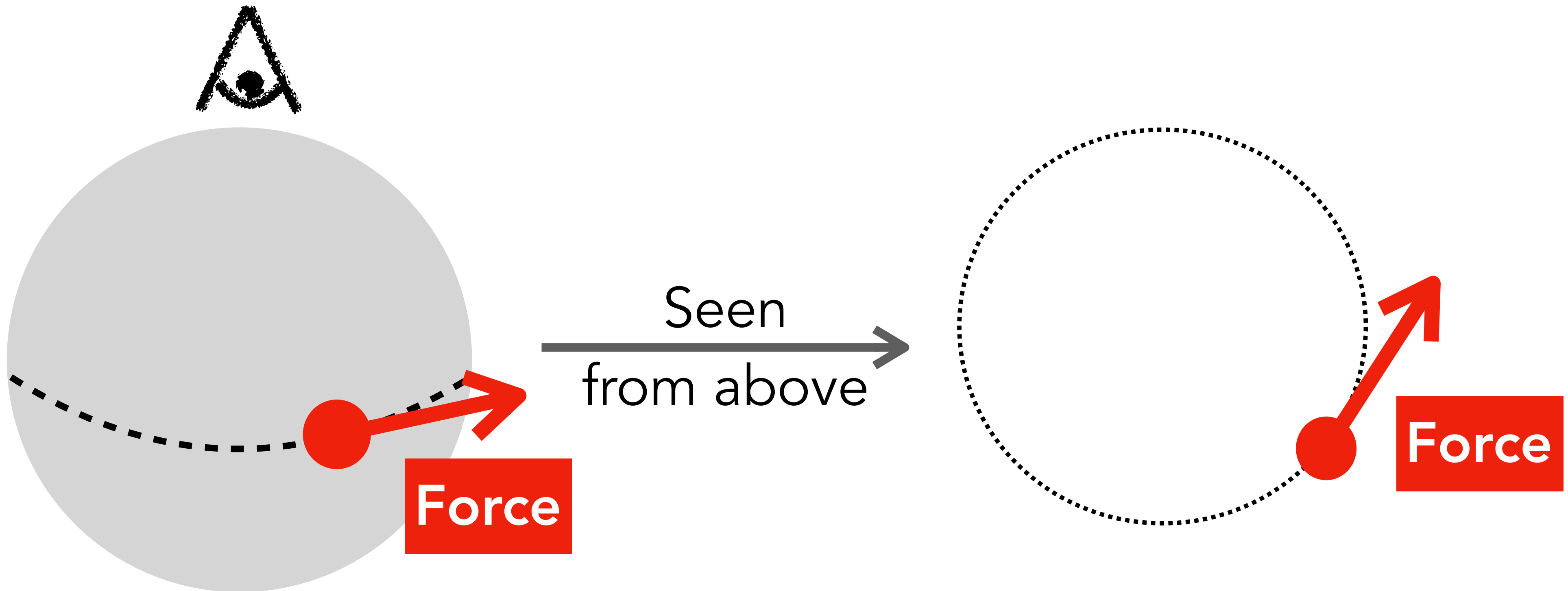
Advance a star



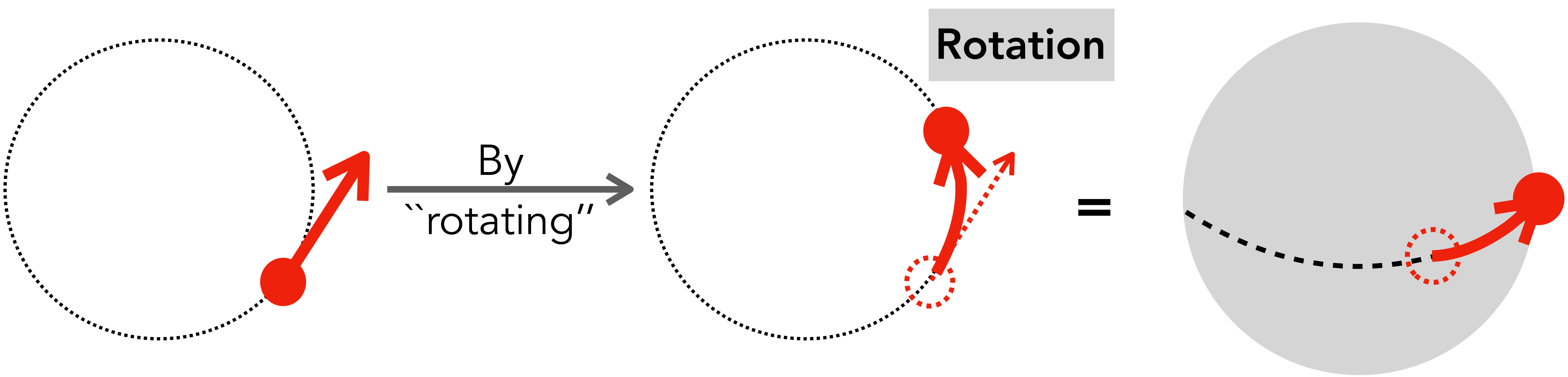
Stars leave the unit sphere. Bad

Classical integration

Force on a star



Advance a star



Stars stay on the unit sphere. Good

Structure-preserving integration

Dynamics on the **unit sphere**

$$\dot{\mathbf{b}} = \mathbf{B}(\mathbf{b}) \quad \text{with} \quad \mathbf{B}(\mathbf{b})\cdot\mathbf{b} = 0$$

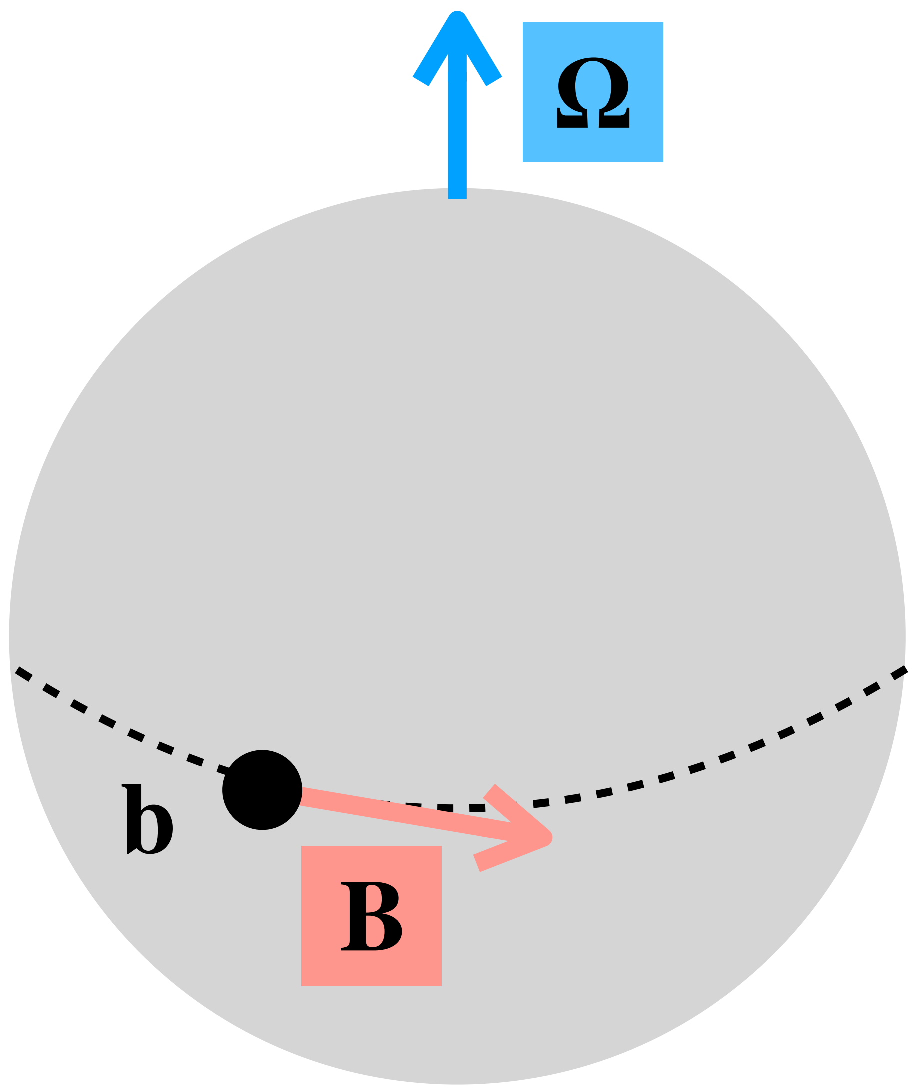
Rotation along **great circle**

$$\dot{\mathbf{b}} = \boldsymbol{\Omega} \times \mathbf{b} \quad \text{with} \quad \boldsymbol{\Omega} = \mathbf{b} \times \dot{\mathbf{b}}$$

Exact solution for fixed $\boldsymbol{\Omega}$

$$\mathbf{b}(t) = \phi[t\,\boldsymbol{\Omega}] \circ \mathbf{b}(0)$$

Rodrigues' rotation formula



Explicit scheme

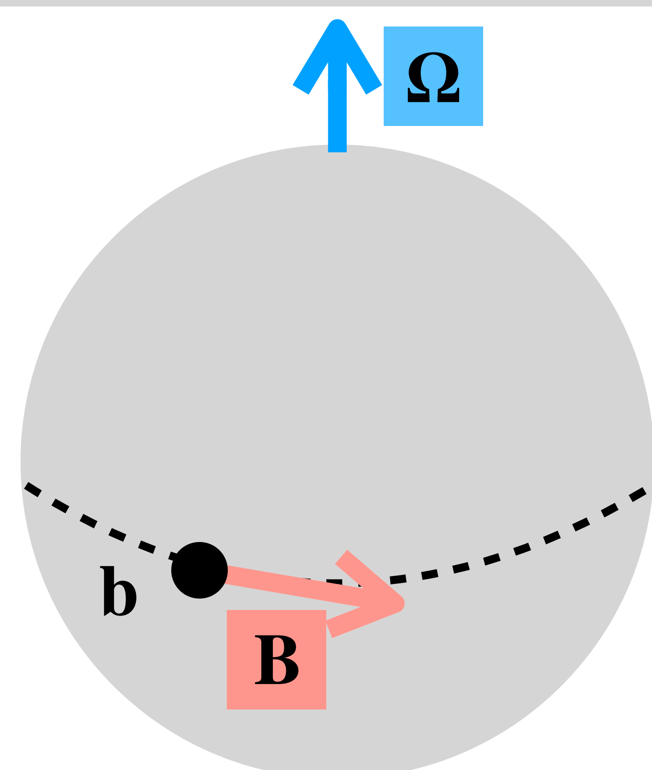
Explicit Midpoint via **rotations**

$$
\begin{aligned}
F_1 &= F(y_n) \\
y_2 &= y_n + \frac{1}{2}\,\tau\, F_1 \\
F_2 &= F(y_2) \\
y_{n+1} &= y_n + \tau\, F_2
\end{aligned}
$$

$\longrightarrow$

MK2

$$
\begin{aligned}
\mathbf{\Omega}_1 &= \mathbf{\Omega}(\mathbf{b}_n) \\
\mathbf{b}_2 &= \phi[\tfrac{1}{2}\tau\,\mathbf{\Omega}_1] \circ \mathbf{b}_n \\
\mathbf{\Omega}_2 &= \mathbf{\Omega}(\mathbf{b}_2) \\
\mathbf{b}_{n+1} &= \phi[\tau \mathbf{\Omega}_2] \circ \mathbf{b}_n
\end{aligned}
$$



Properties:

(i) explicit

(ii) intrinsic

(ii) exactly conserves |b|

(iii) second-order accurate

(iv) two-stage

Adding **commutations**, can be used for high-order schemes *Munthe-Kaas(1999)*

How to do better

Parallelisation

Ladner+(1980)

$$P_{\ell m}(r) = \sum_{j,l;r_{jl}<r} Y_{\ell m}(\mathbf{r})$$

Parallel prefix sum

Multi-timesteps

Saha+(1994)

$$\langle H \rangle = \sum_{(i,j)\in A} \langle H_{ij} \rangle + \sum_{(i,j)\in B} \langle H_{ij} \rangle$$

Hamiltonian splitting

Softening

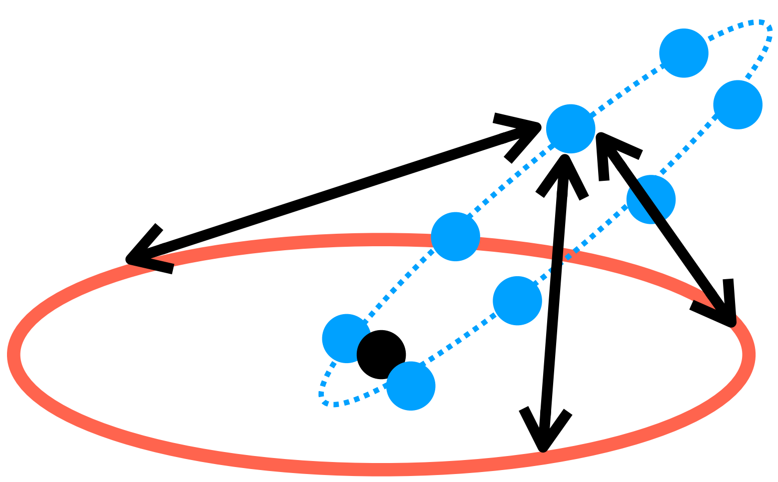
Dehnen+(2014)

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|} \to \varphi_\epsilon(\mathbf{r}-\mathbf{r}')$$

Direct summation & Opening angle

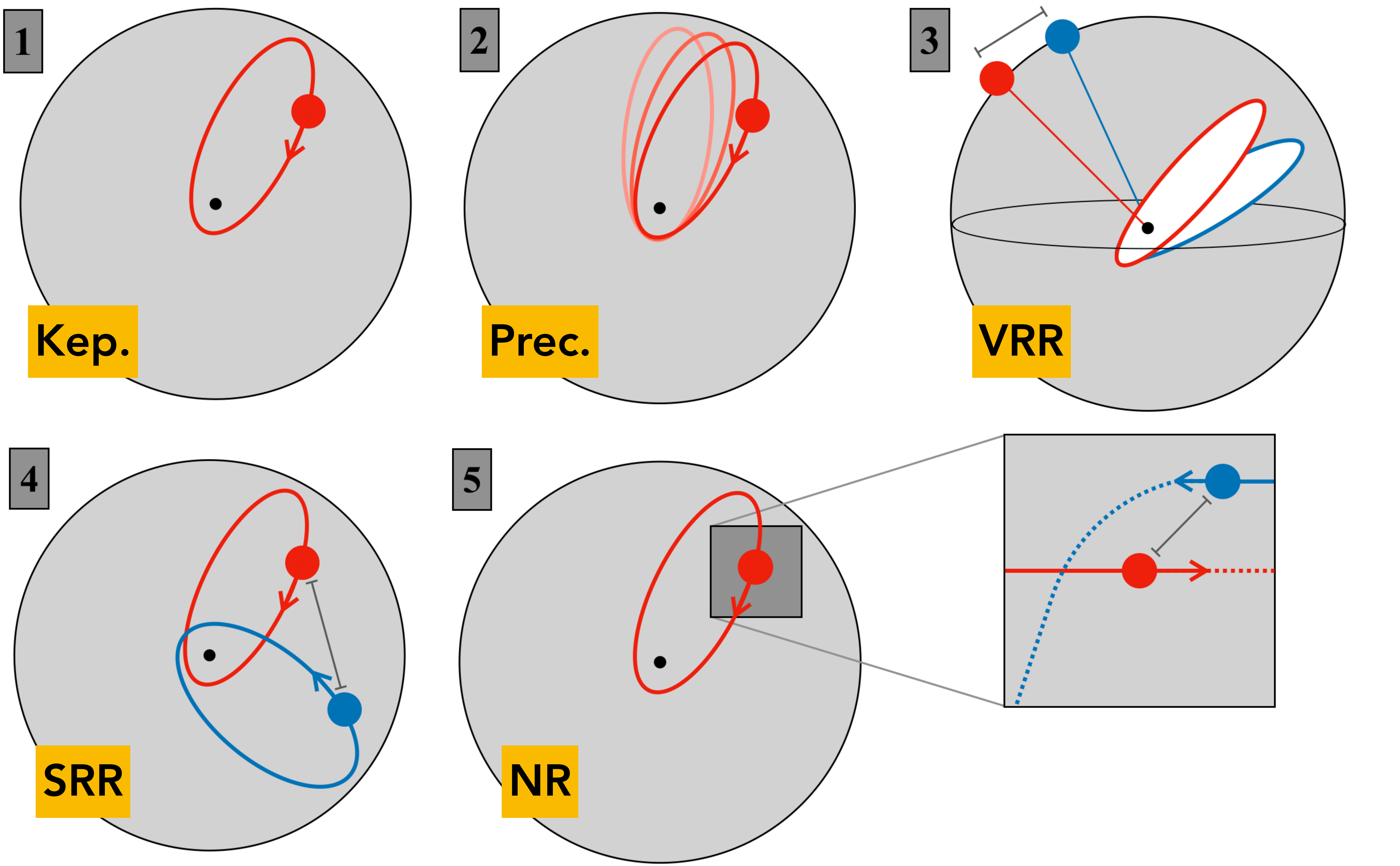
Gauss method

Touma+(2009)



Star-Wire interaction

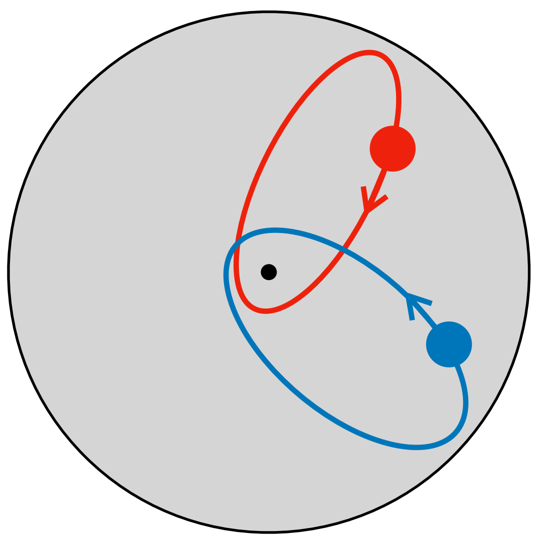
A wealth of dynamical processes



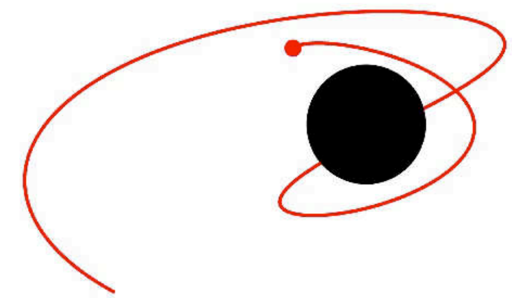
An extremely **hierarchical system**

The future of galactic nuclei

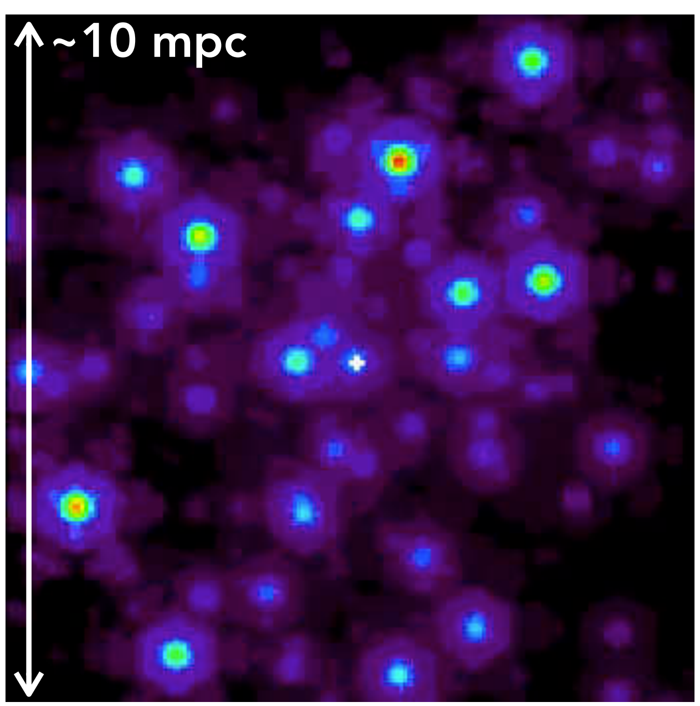
New stellar orbits



Infall of **compact objects**



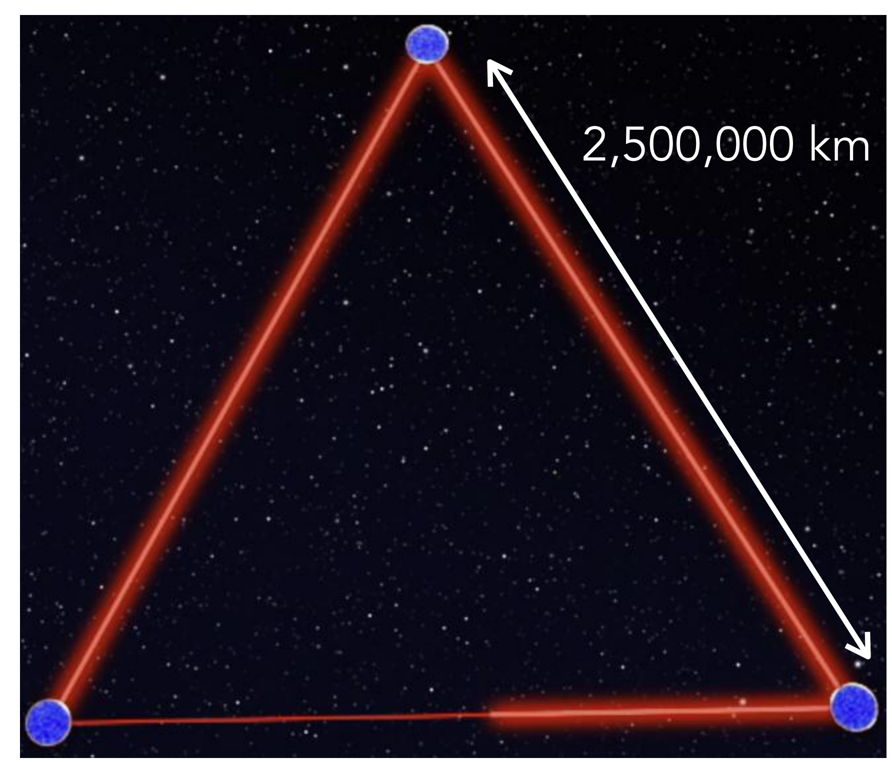
TMT and ELT



UCLA

Expected observations

LISA spatial interferometer



Next steps — Theory & Numerics

Linear response

$$\mathbf{M}(\omega) = \sum_{\mathbf{k}} \int \mathrm{d}\mathbf{J} \frac{G(\mathbf{J})}{\mathbf{k}\cdot\mathbf{\Omega}(\mathbf{J}) - \omega}$$

Response matrix & Modes

Non-axisymmetry

$$F_{\mathrm{tot}} = F_{\mathrm{tot}}(a, e, \widehat{\mathbf{L}})$$

Rotation

More efficient methods

$$T_{\mathrm{Kep}} \propto a^{3/2}$$

$$T_{\mathrm{rel}} \propto a^{4/2}\,(1-e^2)$$

Range of timescales

Time integration

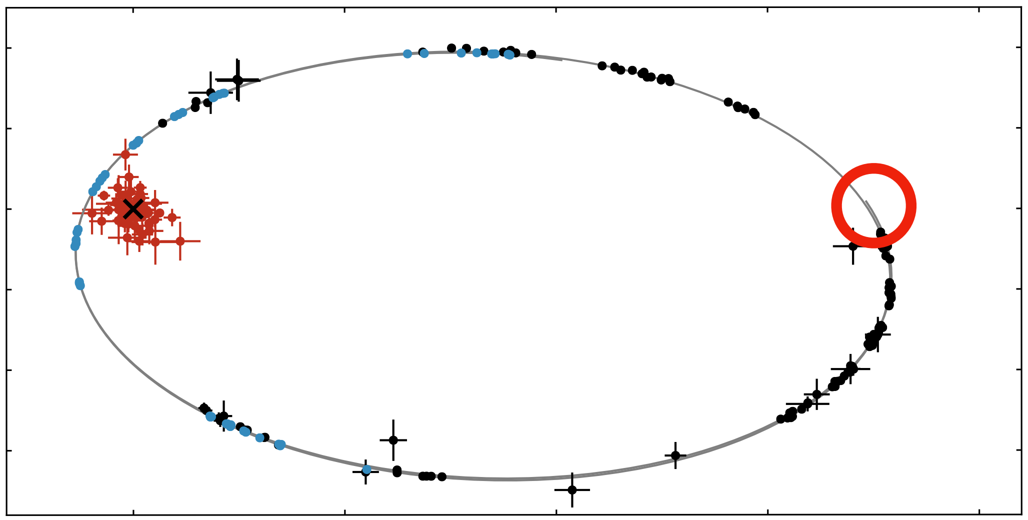
$$\frac{\partial F}{\partial t} = C[F, F]$$

Collision operator

SgrA* is exciting

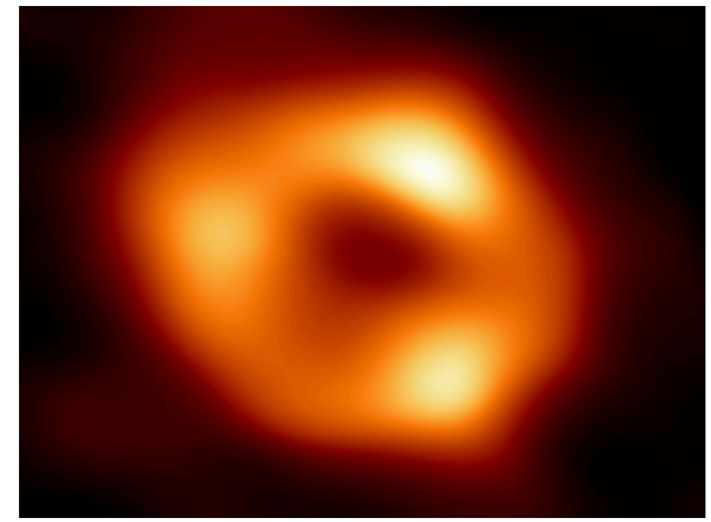
S2’s **relativistic precession**

Gravity+(2020)



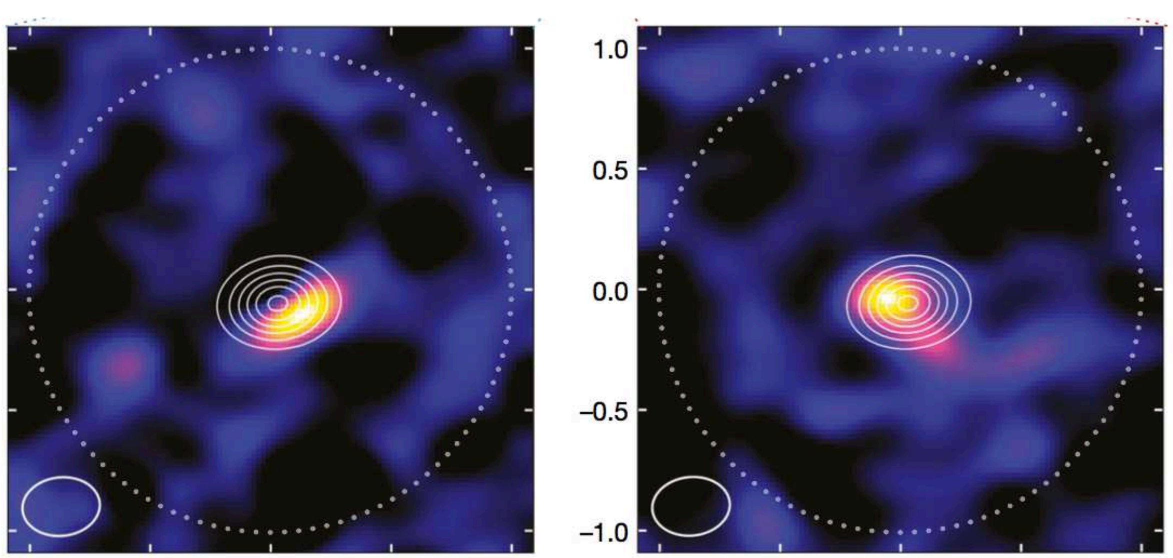
Event Horizon

EHT+(2022)



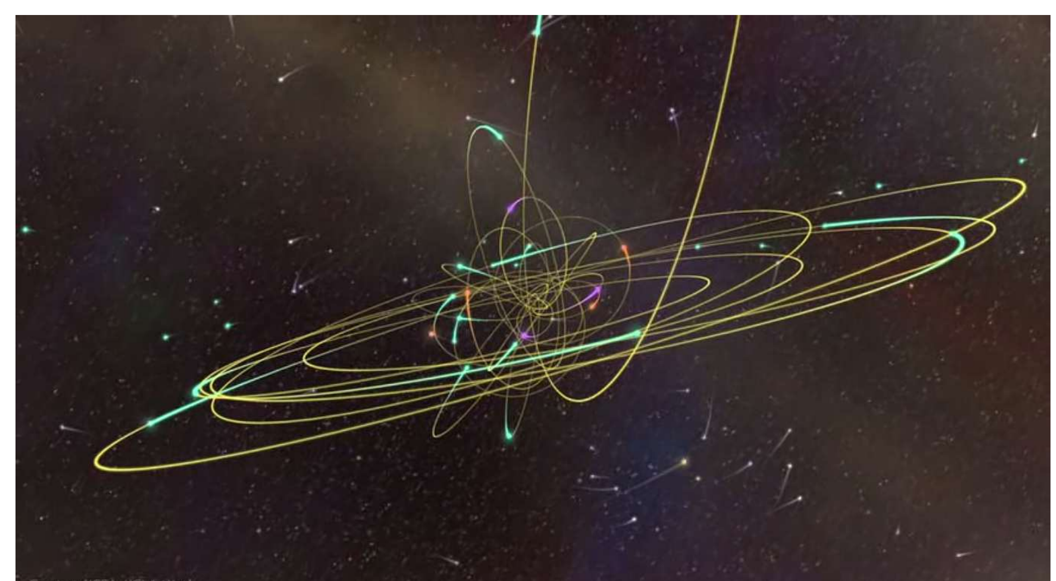
Cold **accretion** disc

Murchikova+(2019)



Clockwise **stellar disc**

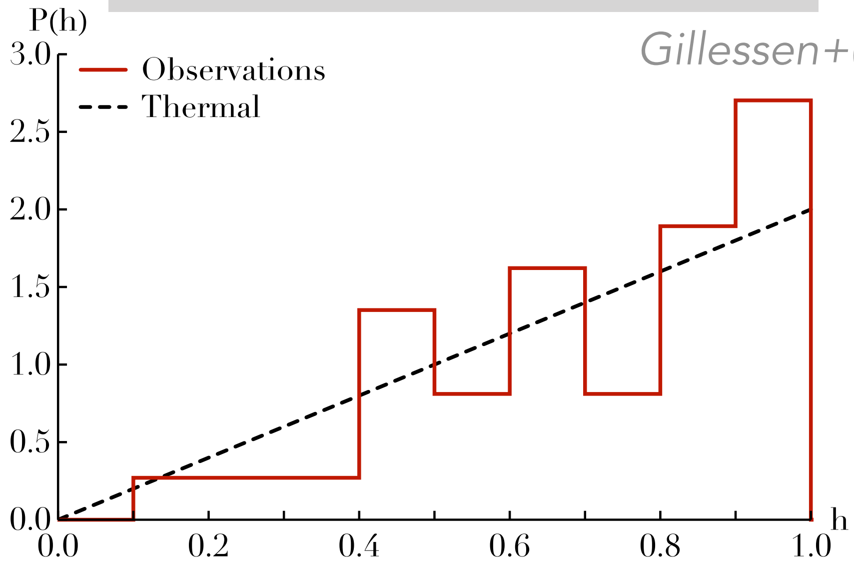
Keck



Next steps — SgrA* & Observations

SRR & Eccentricity

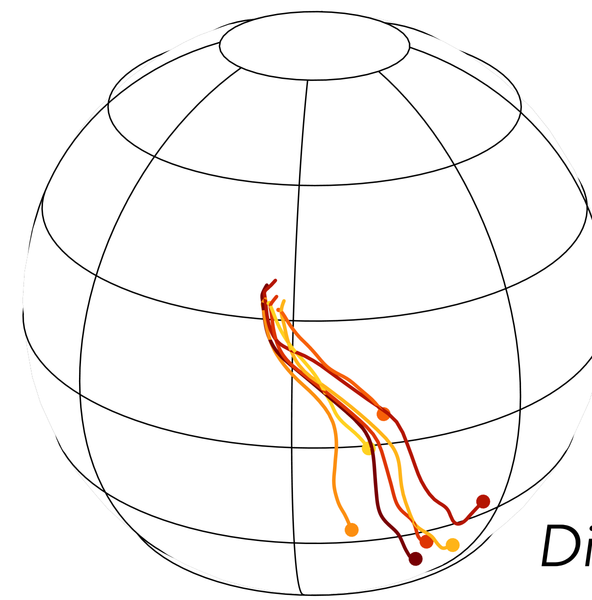
Gillessen+(2017)



Eccentricity thermalisation

VRR & Stellar Discs

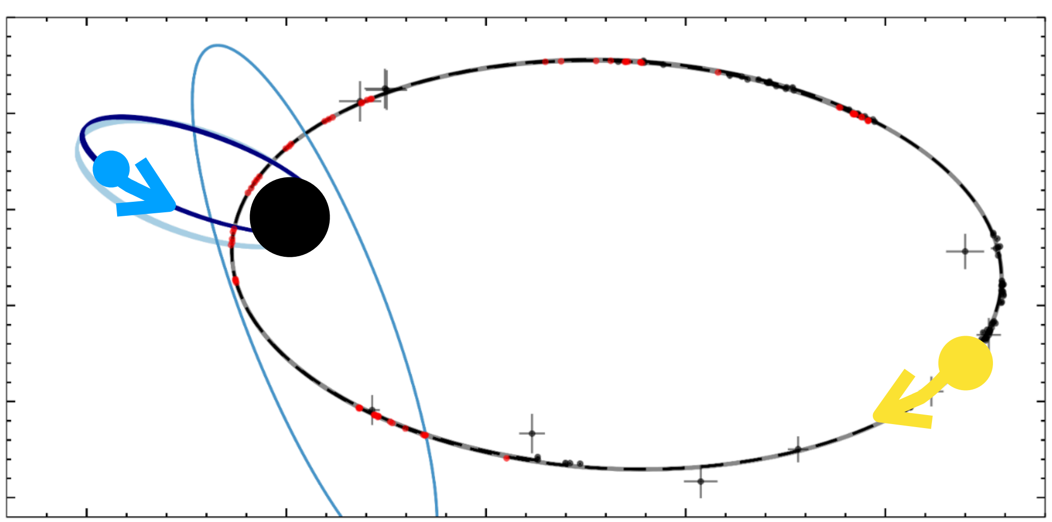
VLT, Keck



Disc dilution

S2's kinematics

Gravity+(2023)
Will+(2023)



Local perturbations?

Future observations

$$P = P(a, e, \widehat{\mathbf{L}})$$

Full PDF statistics

A liquid crystal

Quadrupolar interaction