How to read a caloric curve?

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Consider a self-gravitating system (or any long-range interaction one), with N particles,

K total kinetic energy
W total potential energy.

$2K + W = 3PV$ virial theorem (P, pressure at the edge of volume V, if applicable).

Consider isothermal conditions: $K = \frac{3}{2} N T$ → "temperature"
(assume $k_B = 1$)

E total energy
$E = K + W = -K = -\frac{3}{2} N T < 0$ b/c self gravitating, bounded

$C_V = \frac{dE}{dT} = -\frac{3}{2} N < 0$ negative heat capacity !?
(i.e. by losing energy, the system gets hotter)

Thirring, Lynden-Bell & Wood realised that this is not in contradiction w/ classical stat mech. proof that a system in thermal equilibrium must have positive heat capacity.

Issue: statistical ensemble descriptions of systems with long-range interactions are <u>not</u> equivalent.
Specifically: canonical ≠ microcanonical
CE MCE
↓
allows for negative C_V (corresp. to isothermal phase transitions in CE.)

⊛ Note: formal descriptions; one might argue that the well-posedness of this approach is questionable for physical purposes!

Look for most probable distribution of particles at statistical equilibrium, corresponding to "macrostate" described by $f = f(\underline{r}, \underline{v})$. Assume S Boltzmann. (2

<u>MCE description</u>

$$\text{Max}\left\{ S[f] \,\middle|\, E[f] = E;\ M[f] = M \right\}$$

Conditions of extrema to 1st order:

$$\delta S - \beta \delta E - \alpha \delta M = 0 \qquad \beta, \alpha \text{ Lagrange multipliers}$$

$\rightarrow$ obtain critical points of $S|_{M,E}$

$$f = f(\underline{r}, \underline{v}) = A' e^{-\beta m \left(\frac{v^2}{2} + \Phi(\underline{r}) \right)} \quad (*)$$

β "inverse temperature"

$m = M/N$

α absorbed in normalizing constant A'

$$\rho = \int f(\underline{r}, \underline{v})\, d^3v = $$
$$= A e^{-\beta m \Phi(\underline{r})}$$

note $\Phi(\underline{r}) = \Phi(|\underline{r}|)$

"CALORIC CURVE": $\beta(E)$

needs E, which needs Φ (for fully self-consistency), via

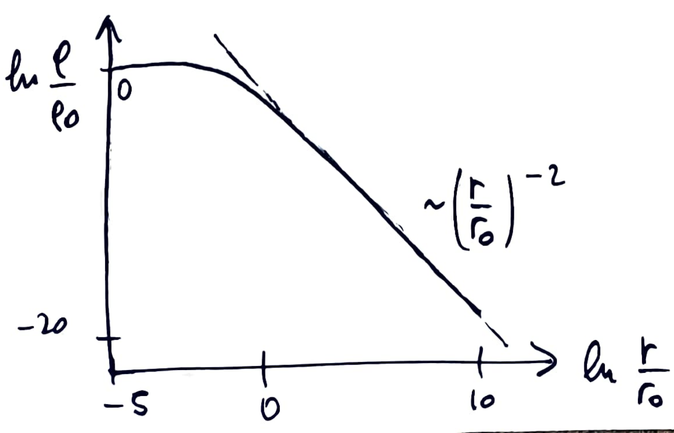
$$\nabla^2 \Phi = 4\pi G \rho = 4\pi G A e^{-\beta m \Phi(r)}.$$

ODE + BCs.

$\rightarrow$ series of equilibria "isothermal spheres"

- regular equilibria (solved numerically)
- singular equilibrium (singular isothermal sphere).

Important role!



$r_0 = 4\pi G \beta m \rho_0$

$\rho_0 = \rho(0)$

<u>Note</u>: "homologous" profiles, via rescaling for ρ_0

<u>Note</u>: regular profiles oscillates around decay!

CE. description

$$\text{Max}\left\{ J[f] = S[f] - \beta E[f] \;\middle|\; M[f] = M \right\}$$

↓ free energy

Conditions of extrema to 1st order:

$\delta J - \alpha\, \delta M = 0$ $\qquad\qquad$ α Lagrange multiplier

→ obtain critical points of $J|_M$.

Find same solutions as (*) ; Maxwell-Boltzmann distribution, i.e. critical points formally coincides (at least at the level of 1st ord. variation).

__But__ 1st order variations of the relevant thermodynamical potential are __not__ sufficient to assess stability of the equilibria, hence determination of the sign of the 2nd order variations is needed.

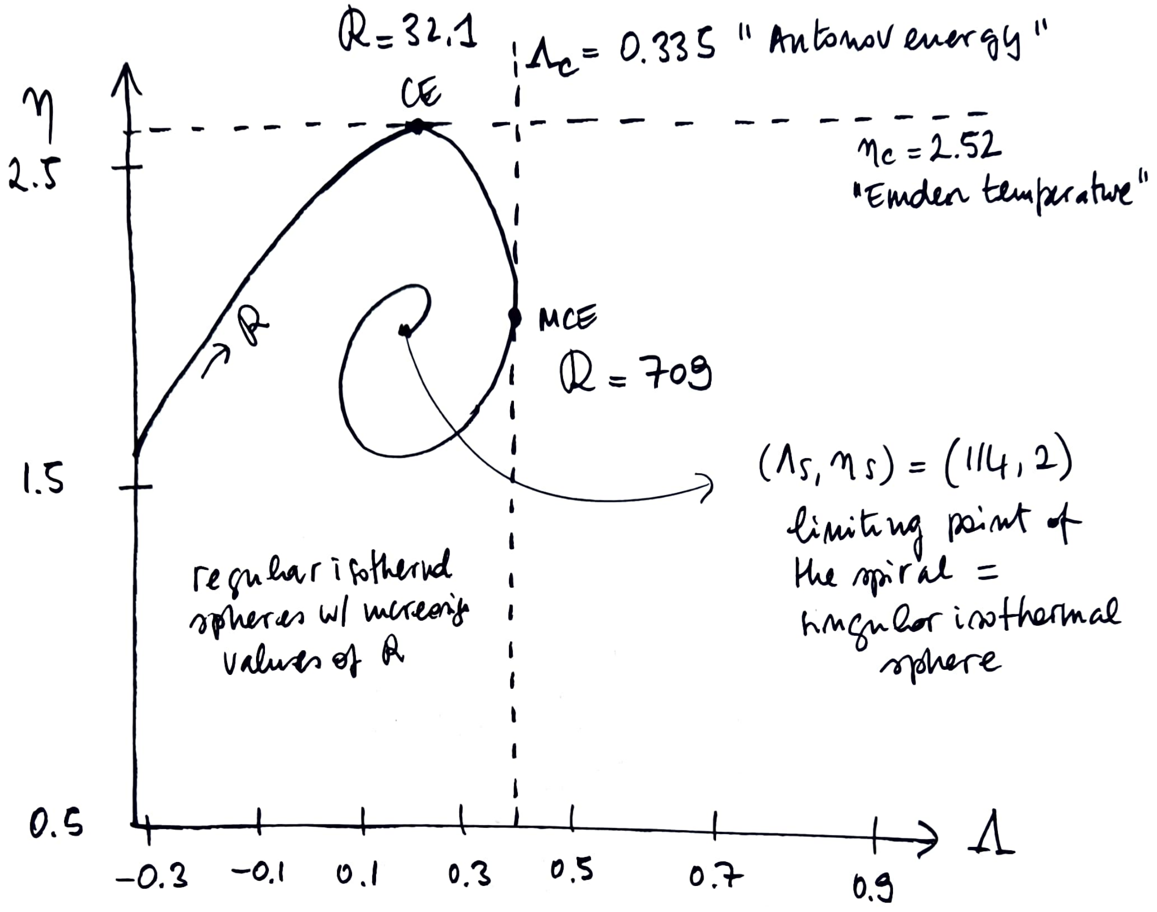
→ look for maxima of $S|_{E,M}$; $J|_M$.

via "turning points method" of Poincaré, i.e. look for turning points in the series of equilibria which tells us plenty about stability conditions in an almost "pictorial" way.

Will realise that differences between MCE / CE descriptions arise naturally.

__Note__ Keep in mind that these are __local__ maxima! No global maximum entropy state exists for S or J (see Scott Tremaine's impromptu), for any self-gravitating system of particle w/o small cut-off, classically described.

"Caloric curve" for isothermal spheres



$$\eta = \frac{\beta G M m}{R}$$ "dimensionless inverse temperature"

$$\Lambda = -\frac{ER}{GM^2}$$ "dimensionless total energy"

<u>Note</u>: R radius of spherical container, introduced for convenience (more details later)

commonly parametrised via "density contrast" $\mathcal{R} = \frac{\rho(0)}{\rho(R)}$

No states available for $\Lambda > \Lambda_c$ (in MCE description)

No states available for $\eta > \eta_c$ (in CE description).

In essence, need sufficiently large energy and temperature

"Turning points method"

[apply first to MCE description]

- States determined via max. process of $S|_{M,E}$; parameter conjugate to S wrt. E is $\beta = \left.\frac{\partial S}{\partial E}\right|_{N,V}$.

  Plot $\beta = \beta(-E)$, interpret curve by two principles:

  (i) a change of stability occurs only at turning points where E is an <u>extremum</u> (i.e. $\frac{d\beta}{dE}$ is infinite).

  (ii) a mode of stability is <u>lost</u> if the curve rotates <u>clockwise</u>, and it is <u>gained</u> if the curve rotates <u>anticlockwise</u>.

In the plot, for Λ (i.e. E) sufficiently large, states are stable [not really, more later]. Physically, this is because the gravity is almost negligible (like an ordinary gas in a box).

LEM = local entropy maxima

all states are stable until point "MCE" (w/ density contrast $\mathcal{R} < 709$).

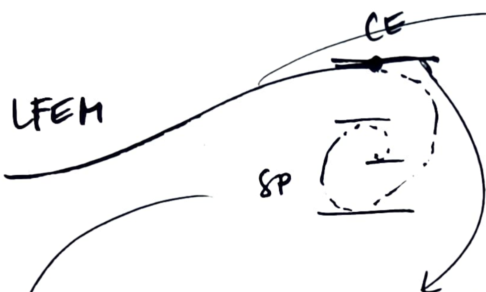
vertical tangent
$\Rightarrow$ one mode of stability is lost.

as the curve spirals inward,
at each tangent point a mode of stability is lost
$\Rightarrow$ states are unstable saddle points (SP) of the entropy until state (Λ_S, m_S), which is the most unstable one, corresponding to the singular isothermal sphere.

[now apply method to CE description]

- States determined via max procen of $J|_M$; parameter conjugate to J wrt β is $-E = \frac{\partial J}{\partial \beta}\Big|_{N,V}$.

To visually apply the two previous principles, must rotate
Note plot by 90° and interpret $E = E(\beta)$, then assess the derivatives.



LFEM = local free energy maxima

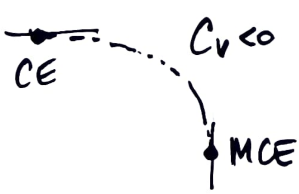
all states are stable until point "CE" (w/ density contrast $\mathcal{R} < 32.1$).

vertical (if rotated), i.e. horizontal, tangent
$\Rightarrow$ one mode of stability is lost.

as the curve spirals inward, at each tangent point a mode of stability is lost (as before)

$\Rightarrow$ states are unstable saddle points (SP) of the free energy, until state (Λ_s, η_s).

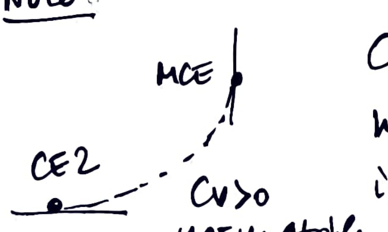
<u>Note</u>:



States between "CE" and "MCE" are <u>stable</u> in MCE description and <u>unstable</u> in the CE description.
These are the states with negative heat capacity! $C_V = \frac{dE}{dT}\Big|_{N,V} < 0$

$\rightarrow$ the ensembles are not equivalent, as Thirring realised.

<u>Note</u>:



$C_V<0$ is a <u>sufficient</u> but not necessary condition for canonical instability!

Note: the states are actually metastable, with a lifetime $\propto N$. Therefore, for $N \to +\infty$, are de facto stable (see Chavanis 2006 and Lynden Bell & Wood 1968 for computation and comments).

What is the fate of the unstable states?

- gravothermal catastrophe

In MCE description, (meta) stable states exist only for sufficiently large energies $E > E_c = -0.335 \frac{GM^2}{R}$ (Antonov Energy). Below E_c, no statistical equilibrium can be achieved and the system enters a catastrophic collapse.

Physical interpretation: stellar system with developing core-halo structure; if the core's temperature is momentarily higher than the halo's temperature, the core loses heat to the halo. But the core then becomes hotter, and the process can not stop (runaway).

(R is the finite size of the stellar system, no thermodynamic role attributed to the boundary).

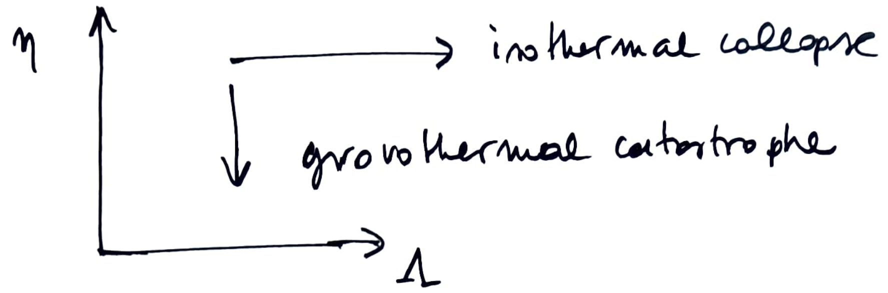
Ultimate product: binary star surrounded by a hot halo.

- Isothermal collapse

In the CE description, (meta) stable states exist only for sufficiently large temperatures $T > T_c = \frac{GMm}{2.52\,R}$ (Emden temperature). Note that, in this description, the system is to be considered in contact with a thermal bath with temperature T. Below T_c, the system can not be in equilibrium with the bath and becomes unstable; it experiences collapse and density increases

Ultimate product: "Dirac peak" (equivalent to Bose-Einstein condensation, in k space).

Evolution in (Λ, η) plane is therefore



Further generalisations

1. different series of equilibria (e.g. truncated isothermal spheres, King, Woolley, Wilson etc.).
2. polytropes, see especially the use of "Milne variables".
3. different statistics (e.g. bosonic or fermionic D.F.).
4. different, generalised entropies.
5. General relativity regime.

References → Chavanis 2006 Int. Jour. Mod Phys.

Antonov 1962, Vest. Leningrad Univ.
Lynden-Bell & Wood 1968 MNRAS
Thirring 1970 Zhy P.
Katz 1978, 1980 MNRAS
} classic studies

→ very readable!

Chavanis 2003 A&A [2]
Chavanis, Lemou, Méhats Part I, II 2015 PRD [3]

— Textbooks.

B&T 2008, Sect 7.3
H&H 2003, Sect 17.
Chandrasekhar 1967 (Milne variables).