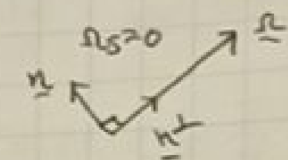
Nonlinear wave-particle interactions in galaxies (KITP, 28th June 2024)

- Inhomogeneous $f_0 = f_0(\underline{J})$
- 2D $\underline{\theta} = (\theta_\varphi, \theta_r)$, $\underline{J} = (J_\varphi, J_r)$
- No collective effects
- Large (nonlinear) external perturbation



Slow–fast angle actions

$$\underbrace{n_\varphi\Omega_\varphi + n_r\Omega_r}_{\underline{n}\cdot\underline{\omega}} - \underbrace{n_\varphi\Omega_p}_{\omega}$$

$$\Omega_s \equiv \underline{n}\cdot\underline{\Omega} \qquad \Omega_f = \underline{n}^\perp\cdot\underline{\Omega} \qquad \left(\begin{array}{l}\underline{\Omega} \equiv (\Omega_\varphi - \Omega_p, \Omega_r) \\ \underline{n} \equiv (n_\varphi, n_r) \\ \underline{n}^\perp \equiv (n_r, -n_\varphi)\end{array}\right)$$

$$\left\{\begin{array}{ll} \theta_s = \underline{n}\cdot\underline{\theta} & \theta_f = \underline{n}^\perp\cdot\underline{\theta} \\ J_s = \dfrac{\underline{n}\cdot\underline{J}}{\underline{n}^2} & J_f = \dfrac{\underline{n}^\perp\cdot\underline{J}}{\underline{n}^2} \end{array}\right.$$

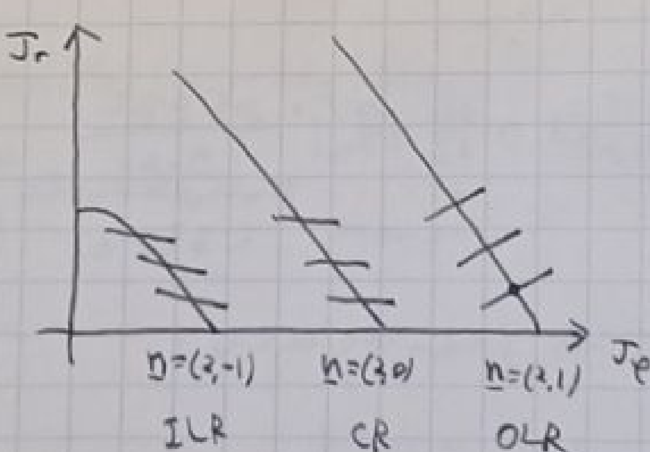
$$S(\underline{\theta}, \underline{J}') = (\underline{n}^\perp\cdot\underline{\theta})J_f + (\underline{n}\cdot\underline{\theta})J_s$$

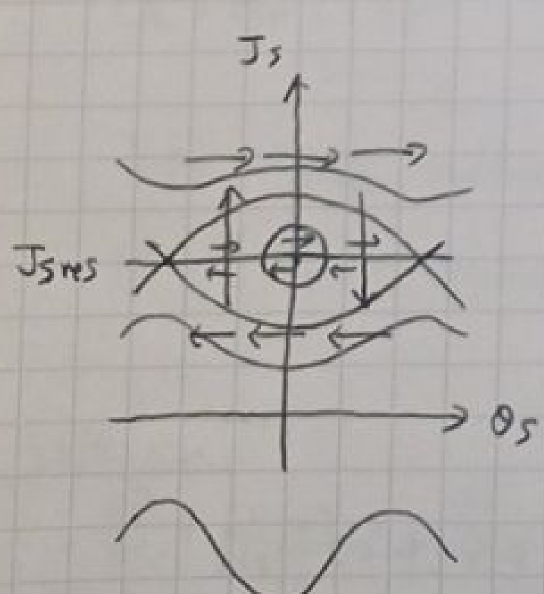
$$\left.\begin{array}{l} J_\varphi = \dfrac{\partial S}{\partial\theta_\varphi} = n_r J_f + n_\varphi J_s \\ J_r = \dfrac{\partial S}{\partial\theta_r} = -n_\varphi J_f + n_r J_s \end{array}\right\rangle \begin{array}{l} n_\varphi J_\varphi + n_r J_r = (n_\varphi^2 + n_r^2)J_s \to J_s = \dfrac{\underline{n}\cdot\underline{J}}{\underline{n}^2} \\ n_r J_\varphi - n_\varphi J_r = (n_r^2 + n_\varphi^2)J_f \to J_f = \dfrac{\underline{n}^\perp\cdot\underline{J}}{\underline{n}^2} \end{array}$$

$$H' = H + \frac{\partial S}{\partial t} = H - n_r\Omega_p J_f - n_\varphi\Omega_p J_s = H - \Omega_p(n_r J_f + n_\varphi J_s) = H - \Omega_p J_f$$

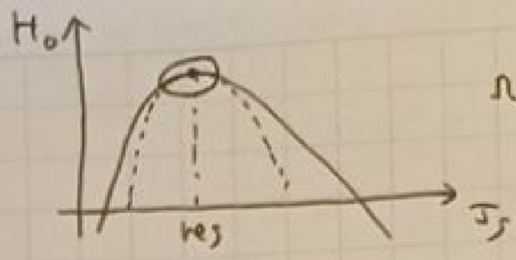
$$\bar{H} = \int\frac{d\theta_f}{2\pi} H$$

$$\dot{J}_f = -\frac{\partial\bar{H}}{\partial\theta_f} = 0$$





$$H = H_0(\underline{J}) + H_1(\theta_s, \underline{J}, t)$$
$$\approx \tfrac{1}{2} G (J_s - J_{s,\mathrm{res}})^2 - F \cos\theta_s$$



$$\Omega_s = \left.\frac{\partial H_0}{\partial J_s}\right|_{\mathrm{res}} = 0$$

$$G = \left.\frac{\partial^2 H_0}{\partial J_s^2}\right|_{\mathrm{res}} = \left.\frac{\partial \Omega_s}{\partial J_s}\right|_{\mathrm{res}} = n_i n_j \frac{\partial \Omega_i}{\partial J_j} \quad (i, j = \varphi, R)$$

$$\left(\begin{aligned} H_1 &= \sum_{\underline{m}} \hat{\Phi}_{\underline{m}}(\underline{J}, t)\, e^{i\underline{m}\cdot\underline{\theta}} \longleftarrow m_s\theta_s + m_f\theta_f \\ &= \sum_{m_s} \hat{\Phi}_{m_s}(\underline{J}, t)\, e^{i m_s \theta_s} \\ &= 2 \sum_{m_s} |\hat{\Phi}_{m_s}| \cos(m_s\theta_s + \varphi_{m_s}) \end{aligned} \right)$$

(<u>Linear theory</u>)

$$f_1(\underline{\theta}, \underline{J}, t) = \sum_{\underline{n}} i\underline{n}\cdot\frac{\partial f_0}{\partial \underline{J}}\, e^{i\underline{n}\cdot\underline{\theta}} \int_0^t dt'\, e^{-i\underline{n}\cdot\underline{\Omega}(t-t')}\, \hat{\Phi}_{\underline{n}}(\underline{J}, t') \longleftarrow \text{time dependence of amplitude}$$

$$= \sum_{\underline{n}} i \frac{\partial f_0}{\partial J_s}\, e^{i\theta_s} \int_0^t dt'\, e^{-i\Omega_s(t-t')}\, \hat{\Phi}_{\underline{n}}(\underline{J}, \cancel{t'})$$

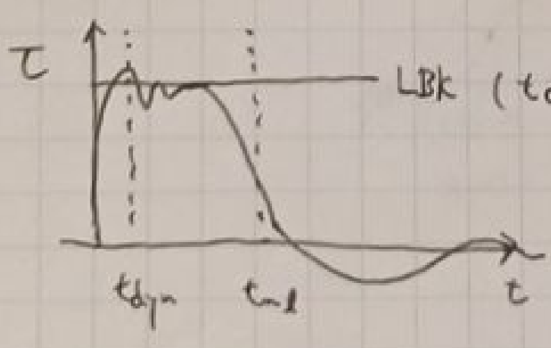
$$= \quad '' \quad \hat{\Phi}_{\underline{n}} \frac{\quad - e^{-i\Omega_s t}}{i\Omega_s}$$

$$= \sum_{\underline{n}} \frac{\partial f_0}{\partial J_s} \hat{\Phi}_{\underline{n}} \frac{e^{i\theta_s} - e^{i(\theta_s - \Omega_s t)}}{\Omega_s}$$

Dynamical friction

lin: $$\tau = (2\pi)^3 \sum_{\underline{n}} n_\varphi \int d^3\underline{J}\, \underline{n}\cdot\frac{\partial f_0}{\partial \underline{J}} |\hat{\Phi}_{\underline{n}}|^2 \underbrace{\frac{\sin \underline{n}\cdot\underline{\Omega} t}{\underline{n}\cdot\underline{\Omega}}}_{\pi\delta(\underline{n}\cdot\underline{\Omega})}$$

$t \to \infty$ $\quad$ (Lynden-Bell Kalnajs formula)



non lin: $f(\theta_s, J_s, t) = f(\theta_\ell, J_\ell, t) = f(\theta_\ell - \Omega_\ell t, J_\ell, t=0)$

LBk homogeneous: $$M \dot{\underline{v}}_p = (2\pi)^3 \int d^3\underline{k}\, \underline{k}\, |\hat{\Phi}_{\underline{k}}|^2 \int d^3\underline{v}\, \underline{k}\cdot\frac{\partial f_0}{\partial \underline{v}}\, \pi\delta[\underline{k}\cdot(\underline{v} - \underline{v}_p)]$$

$$\& \quad \hat{\Phi}_{\underline{k}} = -\frac{4\pi G M}{(2\pi)^3 k^2} e^{-i\underline{k}\cdot\underline{v}_p t}$$

$$\dot{\underline{v}}_p = 4\pi G^2 M \ln\left(\frac{k_{max}}{k_{min}}\right) \int d^3\underline{v}\, f_0 \frac{\underline{v} - \underline{v}_p}{|\underline{v} - \underline{v}_p|^3}$$

Chandrasekhar $\longrightarrow$ $(M \to m)$

Moving resonance

$$\begin{cases} J_s' = J_s - J_{s,res}(t) \\ \theta_s' = \theta_s \end{cases}$$

$$S(\theta_s, J_s', t) = \theta_s \left[J_s' + J_{s,res}(t) \right]$$

$$H' = H + \frac{\partial S}{\partial t}$$

$$= \frac{1}{2} G J_s'^2 - F \cos\theta_s' + \dot{J}_{s,res}\, \theta_s'$$

$$\frac{1}{F} \quad h = \frac{1}{2} j_s^2 \underbrace{- \cos\theta_s + \alpha\, \theta_s}_{V(\theta_s)}$$

$$J_0 = \sqrt{\frac{F}{G}}, \quad \Omega_0 = \sqrt{FG}$$

$$F = J_0 \Omega_0, \quad G = \frac{\Omega_0}{J_0}$$

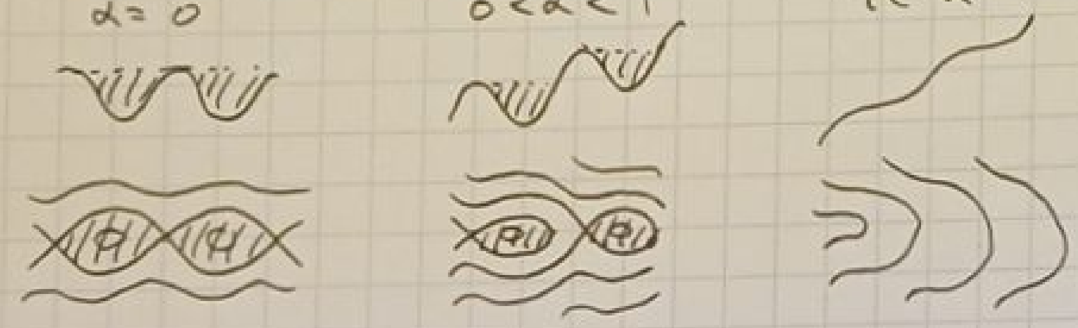
$$j_s = \frac{J_s'}{J_0}, \quad \alpha = \frac{\dot{J}_{s,res}}{J_0 \Omega_0} \sim \frac{T_{lib}}{T_{drift}}$$

$$= \frac{2}{\pi} \frac{T_{lib}}{T_{drift}}$$

$\alpha = 0$ $\quad$ $0 < \alpha < 1$ $\quad$ $1 < \alpha$

$V(\theta_s)$

$h(\theta_s, J_s)$



$$J_s = \frac{\Omega_s}{G}$$

$$j_s = \frac{\Omega_s}{\Omega_0}$$

$$\tau = t\,\Omega_0$$

$$\Delta = D \sqrt{\frac{G}{F^3}}$$

Dynamical feedback

$$\tau_{fb} = (2\pi)^3\, \dot{\Omega}_p \sum_{\underline{n}} n_\ell \int d^3\underline{J}\ \mathrm{sgn}(G)\, (M_0 - M_\ell)\, \delta(\underline{n} \cdot \underline{\Omega})$$

$$M_\ell = \int_0^{J_{\ell, sep}} dJ_\ell\ f(J_f, J_s, t), \quad M_0 = \int_0^{J_{\ell, sep}} dJ_\ell\ f$$

$$\tau_{fb} = \dot{\Omega}_p\, C(\dot{\Omega}_p)$$

$$I \dot{\Omega}_p = \tau_{friction} + \tau_{feedback}$$

$$(I - C)\, \dot{\Omega}_p = \tau_{friction}$$

$$M \dot{u}_p = F_{friction} + F_{feedback}$$

$$(M - C)\, \dot{u}_p = F_{friction}$$

Diffusion

$$\frac{\partial f}{\partial t} + [f, H] = D \frac{\partial^2 f}{\partial J_s^2}$$

$$\frac{\partial f}{\partial \tau} + j_s \frac{\partial f}{\partial \theta_s} - (\sin\theta_s + \alpha) \frac{\partial f}{\partial j_s} = \Delta \frac{\partial^2 f}{\partial j_s^2}$$

$$- \sin\theta_s \frac{\partial f}{\partial j_s} = \alpha \frac{\partial f}{\partial j_s} + \Delta \frac{\partial^2 f}{\partial j_s^2}$$