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Oscillation center quasilinear theory

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A new formulation of the quasilinear theory of weakly turbulent plasmas is presented, which explicitly separates resonant and nonresonant wave-particle interactions from the outset. This is achieved by making a canonical transformation to "oscillation center variables" before attempting to solve the Vlasov equation. A systematic method of constructing the generating function to any order in the wave amplitude is presented, based on a variant of Hamilton–Jacobi perturbation theory. Momentum and energy split naturally into a wave and a particle component. The results are generalized to apply to weakly inhomogeneous plasmas, and verified by demonstrating momentum and energy conservation.

I. INTRODUCTION

Since its introduction,[1,2] there have been two points of view regarding the quasilinear theory of weak plasma turbulence. The first[1,2] is to derive the theory more or less rigorously from the Vlasov equation, and the second[1] is to proceed more heuristically from a quantum mechanical picture in which the plasma waves are quantized (plasmons), and the basic process in quasilinear theory is taken to be absorption or emission of a plasmon by a particle.

The two methods are in agreement on the resonant part of the diffusion tensor (in the classical limit, of course). However, the two methods differ[3] as to their prediction of the effect of nonresonant waves on the (space or ensemble) average particle distribution function $\langle f(\mathbf{x}, \mathbf{p}, t)\rangle$. In the equation for $\langle f\rangle$ derived from the Vlasov equation, the nonresonant contribution can be integrated explicitly,[4] and yields the broadening effect on $\langle f\rangle$ of the "sloshing[5]" of the particles back and forth. In the quantum approach, no such broadening effect is included. Although the sloshing effect is small (proportional to the square of the wave amplitude), and nonsecular (as opposed to diffusion), it is essential to include it when taking moments of the quasilinear diffusion equation (in particular, to show that it conserves energy and momentum).

Nevertheless, the moments of the quantum diffusion equation look "right" in the sense that they assign to the plasmons their expected momenta $\hbar\mathbf{k}$, and energies $\hbar\omega_k$. The heuristic appeal of the quantum picture is such that one wonders if the conservation laws cannot be saved by a *reinterpretation* of the background particle distribution function such that the sloshing component (which, after all, belongs to the waves) is taken out. The author's previous work[6,7] suggests that this can be achieved by a canonical formalism in which the background distribution is defined not as the space average of the exact distribution, but as the distribution corresponding to a fictitious background plasma whose particles move along the average, or oscillation center orbits of the real particles. In this previous work, however, there was no way of treating resonant wave-particle interaction, and it is the aim of the present work to overcome this limitation by including the stochastic motion of the oscillation centers, and the Landau growth or damping of the waves.

In Sec. II we present a canonical transformation to oscillation center variables, such that the interaction Hamiltonian between particles and the wave fields is reduced to an essentially resonant part, while the nonresonant interaction appears as a second-order renormalization of the unperturbed Hamiltonian,[8] (i.e., the nonfluctuating part of the Hamiltonian). In Sec. III we work out the theory in more detail for the usual example of a nonrelativistic plasma interacting via purely electrostatic forces, while in Sec. IV we show how our formulation reconciles the classical and quantum versions of the conservation relations. (Our method is purely classical, but the diffusion equation for oscillation centers corresponds to the classical limit of the quantum diffusion equation.) We also demonstrate that the prescription given in Ref. 1 for generalizing the quasilinear diffusion equation to weakly inhomogeneous systems, namely, replacing $\partial f_0/\partial t$ by $\partial f_0/\partial t + [f_0, H_0]$, can only be correct if H_0 includes the renormalization energy. In view of current interest in anomalous transport in inhomogeneous systems, this may be the most important contribution of the present work.

II. THE GENERAL METHOD

We shall use the standard type of generating function $F_2(\mathbf{x}, \mathbf{P}, t)$, in the notation of Goldstein.[9] By definition, a particle can never be very far from its oscillation center, and the transformation is therefore close to an identity transformation

$$F_2(\mathbf{x}, \mathbf{P}, t) \equiv \mathbf{x}\cdot\mathbf{P} + S(\mathbf{x}, \mathbf{P}, t), \tag{1}$$

where S is required to be nonsecular.[10] Since we are discussing weak turbulence, we order S to be $O(\lambda)$, where λ is a smallness parameter characterizing the wave amplitude. The transformation equations[9] are

$$\mathbf{X}=\mathbf{x}+\frac{\partial S(\mathbf{x},\mathbf{P},t)}{\partial \mathbf{P}} \tag{2}$$

$$\mathbf{p}=\mathbf{P}+\frac{\partial S(\mathbf{x},\mathbf{P},t)}{\partial \mathbf{x}}, \tag{3}$$

$$K(\mathbf{X},\mathbf{P},t)=H(\mathbf{x},\mathbf{p},t)+\frac{\partial S(\mathbf{x},\mathbf{P},t)}{\partial t}, \tag{4}$$

where K is the new Hamiltonian.

Eliminating $\mathbf{X}$ and $\mathbf{p}$ from Eq. (4) by use of Eqs. (2) and (3), we have

$$\frac{\partial S(\mathbf{x},\mathbf{P},t)}{\partial t}+H\left(\mathbf{x},\mathbf{P}+\frac{\partial S}{\partial \mathbf{x}},t\right)-K\left(\mathbf{x}+\frac{\partial S}{\partial \mathbf{P}},\mathbf{P},t\right)=0. \tag{5}$$

If we were to set $K=0$, we would have the Hamilton–Jacobi equation, $\mathbf{X}$ and $\mathbf{P}$ being constants of the motion. However, S would be secular. To keep the transformation close to the identity, we require that the local time average of S following a characteristic (i.e., a particle trajectory in $\mathbf{x}$, $\mathbf{P}$ space) be zero.

Write

$$H=H_0+H_1, \qquad K=H_0+K', \tag{6}$$

where H_0 is the unperturbed Hamiltonian, and H_1 is the interaction Hamiltonian between the particle and the wave fields, and is therefore $O(\lambda)$. The correction K' contains both the $O(\lambda)$ residual interaction, and an $O(\lambda^2)$ "energy level shift." Expanding Eq. (5), then, we have

$$\frac{\partial S}{\partial t'}+\frac{\partial H_0}{\partial \mathbf{p}}\cdot\frac{\partial S}{\partial \mathbf{x}}-\frac{\partial H_0}{\partial \mathbf{x}}\cdot\frac{\partial S}{\partial \mathbf{p}}=K'+\frac{\partial K'}{\partial \mathbf{x}}\cdot\frac{\partial S}{\partial \mathbf{p}}-H_1$$
$$-\frac{\partial H_1}{\partial \mathbf{p}}\cdot\frac{\partial S}{\partial \mathbf{x}}+\frac{1}{2}\frac{\partial^2 H_0}{\partial \mathbf{x}\,\partial \mathbf{x}}:\frac{\partial S}{\partial \mathbf{p}}\frac{\partial S}{\partial \mathbf{p}}-\frac{1}{2}\frac{\partial^2 H_0}{\partial \mathbf{p}\,\partial \mathbf{p}}:\frac{\partial S}{\partial \mathbf{x}}\frac{\partial S}{\partial \mathbf{x}}+O(\lambda^3), \tag{7}$$

where we have now chosen to denote the independent momentum variable by $\mathbf{p}$ rather than $\mathbf{P}$, so that S denotes $S(\mathbf{x},\mathbf{p},t')$, because we wish to integrate Eq. (7) along the unperturbed orbits $\mathbf{x}_0(t'\mid \mathbf{x},\mathbf{p},t)$, $\mathbf{p}_0(t'\mid \mathbf{x},\mathbf{p},t)$ corresponding to the Hamiltonian $H_0(\mathbf{x}_0, \mathbf{p}_0, t')$, and such that $\mathbf{x}_0(t\mid \mathbf{x},\mathbf{p},t)=\mathbf{x}$ and $\mathbf{p}_0(t\mid \mathbf{x}, \mathbf{p},t)=\mathbf{p}$. We can satisfy the requirement that S average to zero on a trajectory by defining K' by

$$K'(\mathbf{x},\mathbf{p},t)\equiv A\left(H_1+\frac{\partial H_1}{\partial \mathbf{p}}\cdot\frac{\partial S}{\partial \mathbf{x}}-\frac{\partial K'}{\partial \mathbf{x}}\cdot\frac{\partial S}{\partial \mathbf{p}}\right.$$
$$\left.+\frac{1}{2}\frac{\partial^2 H_0}{\partial \mathbf{p}\,\partial \mathbf{p}}:\frac{\partial S}{\partial \mathbf{x}}\frac{\partial S}{\partial \mathbf{x}}-\frac{1}{2}\frac{\partial^2 H_0}{\partial \mathbf{x}\,\partial \mathbf{x}}:\frac{\partial S}{\partial \mathbf{p}}\frac{\partial S}{\partial \mathbf{p}}+O(\lambda^3)\right), \tag{8}$$

where A is a local time-averaging operator, to be defined, and $\mathbf{x}$ and $\mathbf{p}$ are to be replaced by $\mathbf{x}_0(t'\mid \mathbf{x},\mathbf{p},t)$ and $\mathbf{p}_0(t'\mid \mathbf{x},\mathbf{p},t)$ in the averaging process. We define the averaging operator A by its action on an arbitrary function $f(t)$,

$$Af(t)\equiv\int_{-\infty}^{\infty}dt'\,\Phi(t-t')f(t'), \tag{9}$$

where[11]

$$\Phi(\tau)\equiv(\pi\tau)^{-1}\sin(\tfrac{1}{2}\tau\Delta\omega).$$

More useful is the Fourier transform

$$\Phi(\tau)=\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\Phi(\omega)\exp(-i\omega\tau),$$

where

$$\Phi(\omega)=\theta(\tfrac{1}{2}\Delta\omega-|\omega|), \tag{10}$$

θ being the unit Heaviside step function. We take $\Delta\omega$ to be much less than the width of a typical spectral feature, but much greater than a typical inverse diffusion time.

With the choice of K' in Eq. (8), Eq. (7) becomes

$$\frac{DS}{Dt}=-(1-A)\left(H_1+\frac{\partial H_1}{\partial \mathbf{p}}\cdot\frac{\partial S}{\partial \mathbf{x}}-\frac{\partial K'}{\partial \mathbf{x}}\cdot\frac{\partial S}{\partial \mathbf{p}}\right.$$
$$\left.+\frac{1}{2}\frac{\partial^2 H_0}{\partial \mathbf{p}\,\partial \mathbf{p}}:\frac{\partial S}{\partial \mathbf{x}}\frac{\partial S}{\partial \mathbf{x}}-\frac{1}{2}\frac{\partial^2 H_0}{\partial \mathbf{x}\,\partial \mathbf{x}}:\frac{\partial S}{\partial \mathbf{p}}\frac{\partial S}{\partial \mathbf{p}}+O(\lambda^3)\right), \tag{11}$$

where D/Dt denotes the convective derivative in phase space. Equations (8) and (11) are to be solved iteratively, thus generating a canonical transformation such that only the *resonant* part of the wave-particle interaction is left in the new interaction Hamiltonian. Having found K', one must then solve the new Vlasov equation

$$\frac{\partial F(\mathbf{x},\mathbf{p},t)}{\partial t}+\frac{\partial K}{\partial \mathbf{p}}\cdot\frac{\partial F}{\partial \mathbf{x}}-\frac{\partial K}{\partial \mathbf{x}}\cdot\frac{\partial F}{\partial \mathbf{p}}=0, \tag{12}$$

where F is the distribution function for oscillation centers. Since the transformation is canonical, F is simply related to the old distribution function f by

$$f(\mathbf{x},\mathbf{p},t)=F(\mathbf{X},\mathbf{P},t), \tag{13}$$

where $\mathbf{X}$ and $\mathbf{P}$ are found by iterating Eqs. (2) and (3)

$$\mathbf{X}(\mathbf{x},\mathbf{p},t)=\mathbf{x}+\frac{\partial S(\mathbf{x},\mathbf{p},t)}{\partial \mathbf{p}}-\frac{\partial S}{\partial \mathbf{x}}\cdot\frac{\partial^2 S}{\partial \mathbf{p}\,\partial \mathbf{p}}+\cdots,$$

$$\mathbf{P}(\mathbf{x},\mathbf{p},t)=\mathbf{p}-\frac{\partial S(\mathbf{x},\mathbf{p},t)}{\partial \mathbf{x}}+\frac{\partial S}{\partial \mathbf{x}}\cdot\frac{\partial^2 S}{\partial \mathbf{p}\,\partial \mathbf{x}}+\cdots. \tag{14}$$

The charge density $\rho(\mathbf{x},t)$ and current density $\mathbf{j}(\mathbf{x},t)$

are then easily computed:

$$\rho(\mathbf{x},t)=\sum_s e_s\int d^3p\, f_s(\mathbf{x},\mathbf{p},t),$$

$$\mathbf{j}(\mathbf{x},t)=\sum_s e_s\int d^3p\,\frac{\partial H_s}{\partial \mathbf{p}} f_s(\mathbf{x},\mathbf{p},t), \qquad (15)$$

where the sum $\sum_s$ is over particle species. Use of Eq. (15) in Maxwell's equations completes the description of the system.

We have completed the schematic description of our transformation method, but it should be remarked here that we have glossed over the central problem of plasma kinetic theory, namely, solving the Vlasov equation (12). The present method has not in any way solved this problem, so we must still specify by what approximation scheme we mean to treat Eq. (12). We shall, in fact, adopt the standard approach of linearizing Eq. (12), and solving an initial value problem for F_1. The asymptotic longtime behavior of Eqs. (15) does not depend on the initial value of F_1, since its contribution decays by phase mixing. Thus, we may discard initial value terms and assume that F_1 consists only of the forced linear response to the wave field. We write (assuming average homogeneity)

$$F(\mathbf{x},\mathbf{p},t)=F_0(\mathbf{p},t)+F_1(\mathbf{x},\mathbf{p},t)+O(\lambda^2), \qquad (16)$$

where F_1 obeys

$$\frac{\partial F_1}{\partial t}+\frac{\partial H_0}{\partial \mathbf{p}}\cdot\frac{\partial F_1}{\partial \mathbf{x}}-\frac{\partial H_0}{\partial \mathbf{x}}\cdot\frac{\partial F_1}{\partial \mathbf{p}}=\frac{\partial K'}{\partial \mathbf{x}}\cdot\frac{\partial F_0}{\partial \mathbf{p}}, \qquad (17)$$

while F_0 obeys the quasilinear diffusion equation

$$\frac{\partial F_0}{\partial t}=\frac{\partial}{\partial \mathbf{p}}\cdot\left\langle\frac{\partial K'}{\partial \mathbf{x}}F_1\right\rangle. \qquad (18)$$

The linear contribution $f_1(\mathbf{x},\mathbf{p},t)$ to the ordinary distribution function is seen from Eqs. (13), (14), and (16) to be given by

$$f_1(\mathbf{x},\mathbf{p},t)=F_1(\mathbf{x},\mathbf{p},t)-\frac{\partial S}{\partial \mathbf{x}}\cdot\frac{\partial F_0}{\partial \mathbf{p}}, \qquad (19)$$

the nonresonant linear response being given by the term involving S, and the resonant linear response by the term involving F_1. Using Eqs. (11), (17), and (18) we may show that f_1 defined by Eq. (19) obeys the ordinary linearized Vlasov equation to $O(\lambda)$, so that the linear response is indeed independent of the breakup between the resonant and nonresonant regions. Thus, having found S there is actually no need to find F_1 by direct calculation, as f_1 may be found by letting the width of the resonant region go to zero and deforming the integration contours in such a fashion as to maintain causality.

III. THE ELECTROSTATIC APPROXIMATION

We shall now confine our attention to nonrelativistic collisionless plasmas with no external magnetic field, and shall consider only electrostatic waves. Thus, we take

$$H_0=(p^2/2m),\qquad H_1=L^{-3}\sum_k e\phi_k(t)\exp(i\mathbf{k}\cdot\mathbf{x}), \qquad (20)$$

where $\sum_k$ sums over all wave vectors $\mathbf{k}$ allowed by periodic boundary conditions ($\phi_0\equiv0$). There has been considerable controversy over the correct way to handle complex frequencies,[12] which we avoid by following Kaufman[4] in the use of the Eikonal representation (in which all frequencies are real):

$$\phi_k(t)=\sum_l \phi_k{}^l(t)\exp[-i\theta_k{}^l(t)],$$

$$\theta_k{}^l(t)\equiv\int_{t_0}^t \omega_k{}^l(t')\,dt' \qquad (21)$$

where $\phi_k{}^l$ and $\omega_k{}^l$ are slowly varying functions of time due to wave growth or damping, and modification of the background distribution by diffusion, respectively. We take diffusion to be the slower process since the diffusion tensor is $O(\lambda^2)$. The superscript l distinguishes both different wave modes and the positive and negative frequency branches of the same mode ($l\gtrless0$ implies $\omega_k{}^l\gtrless0$ in a suitable reference frame). The reality of $\phi_k{}^l$ is insured by the conditions $\omega_{-k}{}^{-l}=-\omega_k{}^l$, $\phi_{-k}{}^{-l}=\phi_k{}^{l*}$.

To $O(\lambda)$ we may show that for the kth Fourier component of K' Eq. (8) gives

$$K_k'=\sum_l e\left(\phi_k{}^l+i\dot\phi_k{}^l\frac{\partial}{\partial\omega_k{}^l}\right)\times\Phi(\omega_k{}^l-\mathbf{k}\cdot\mathbf{p}/m)\exp(-i\theta_k{}^l). \qquad (22)$$

Also Eq. (11) becomes

$$\left(\frac{\partial}{\partial t}+i\mathbf{k}\cdot\mathbf{p}/m\right)S_k=-\sum_l e\left(\phi_k{}^l+i\dot\phi_k{}^l\frac{\partial}{\partial\omega_k{}^l}\right)\times\Pi(\omega_k{}^l-\mathbf{k}\cdot\mathbf{p}/m)\exp(-i\theta_k{}^l), \qquad (23)$$

where

$$\Pi(\omega)\equiv1-\Phi(\omega).$$

We define S in such a way that there is no initial phase information. Thus, Eq. (31) is solved by integrating from $t=-\infty$. We then find

$$S_k=-\sum_l e\left(\phi_k{}^l+i\dot\phi_k{}^l\frac{\partial}{\partial\omega_k{}^l}\right)\left(\frac{i\Pi(\omega_k{}^l-\mathbf{k}\cdot\mathbf{p}/m)}{\omega_k{}^l-\mathbf{k}\cdot\mathbf{p}/m}\right)\times\exp(-i\theta_k{}^l). \qquad (24)$$

We may now evaluate the nonlinear part of K'. The $\mathbf{k}=0$ component of K', which we call the renormalization energy and denote by $\Delta(\mathbf{p})$, is given by

$$\Delta(\mathbf{p})=-\sum_{k,l}\frac{4\pi e^2}{m}\frac{k^2|\phi_k{}^l|^2}{8\pi L^6}\frac{\partial}{\partial\omega_k{}^l}\frac{\Pi(\omega_k{}^l-\mathbf{k}\cdot\mathbf{p}/m)}{\omega_k{}^l-\mathbf{k}\cdot\mathbf{p}/m}. \qquad (25)$$

This represents an extension to the case of resonant interaction of the formula found by the averaged Lagrangian method.[7] It is encouraging to note that $\Delta(\mathbf{p})$ is well defined in the limit $\Delta\omega\to 0$, since $\Pi(\ )$ may then be replaced by the principal part operator P. Because we have, in fact, assumed $\Delta\omega$ to be small compared with the width of the spectrum, the error involved in letting $\Delta\omega\to 0$ is negligible.

The linear response f_1 has the kth Fourier component

$$f_k=-\sum_l \exp(-i\theta_k{}^l)\mathbf{k}\cdot\frac{\partial F_0}{\partial \mathbf{p}}e\left(\phi_k{}^l+i\dot{\phi}_k{}^l\frac{\partial}{\partial\omega_k{}^l}\right)\times\frac{1}{\omega_k{}^l-\mathbf{k}\cdot\mathbf{p}/m+i0}. \quad (26)$$

Inserting this in Eq. (15) gives

$$\rho_k=-\frac{k^2}{4\pi}\left((\epsilon-1)\phi_k{}^l+i\frac{\partial\epsilon}{\partial\omega_k{}^l}\dot{\phi}_k{}^l\right), \quad (27)$$

where

$$\epsilon(\mathbf{k},\omega)\equiv 1+\sum_s\frac{4\pi e^2}{k^2}\int d^3p\,\frac{\mathbf{k}\cdot(\partial F_0/\partial\mathbf{p})}{\omega-\mathbf{k}\cdot\mathbf{p}/m+i0}. \quad (28)$$

Poisson's equation implies

$$\epsilon(\mathbf{k},\omega_k{}^l)\phi_k{}^l+i\frac{\partial\epsilon}{\partial\omega_k{}^l}\dot{\phi}_k{}^l=0. \quad (29)$$

Denote ϵ by $\epsilon_r+i\epsilon_i$, and assume ϵ_i small. Then, we may define $\omega_k{}^l$ consistently with Eq. (29) by

$$\epsilon_r(\mathbf{k},\omega_k{}^l)=0. \quad (30)$$

It follows that

$$\frac{\partial}{\partial t}\mid\phi_k{}^l\mid^2=2\gamma_k{}^l\mid\phi_k{}^l\mid^2, \quad (31)$$

$$\gamma_k{}^l=-\left(\frac{\partial\epsilon_r}{\partial\omega_k{}^l}\right)^{-1}\epsilon_i(\mathbf{k},\omega_k{}^l)$$

$$=\left(\frac{\partial\epsilon_r}{\partial\omega_k{}^l}\right)^{-1}\sum_s 4\pi e^2\int d^3p\,\pi\delta(\omega_k{}^l-\mathbf{k}\cdot\mathbf{p}/m)\,\frac{\mathbf{k}}{k^2}\cdot\frac{\partial F_0}{\partial\mathbf{p}}. \quad (32)$$

Equation (18) gives

$$\frac{\partial F_0}{\partial t}=\frac{\partial}{\partial\mathbf{p}}\cdot\left(\mathsf{D}\cdot\frac{\partial F_0}{\partial\mathbf{p}}\right), \quad (33)$$

where

$$\mathsf{D}=4\pi e^2\sum_{k,l}(\mid\phi_k{}^l\mid^2/8\pi L^6)\mathbf{kk}2\pi\delta(\omega_k{}^l-\mathbf{k}\cdot\mathbf{p}/m) \quad (34)$$

is the *resonant* part only of the quasilinear diffusion coefficient.

IV. MOMENTUM AND ENERGY CONSERVATION IN AN INHOMOGENEOUS PLASMA

We now wish to show that the following generalizations of Eqs. (31) and (33) satisfy momentum and energy conservation. We expect[6,13] Eq. (31) to generalize to

$$\frac{\partial n_k{}^l}{\partial t}+\frac{\partial\omega_k{}^l}{\partial\mathbf{k}}\cdot\frac{\partial n_k{}^l}{\partial\mathbf{x}}-\frac{\partial\omega_k{}^l}{\partial\mathbf{x}}\cdot\frac{\partial n_k{}^l}{\partial\mathbf{k}}=2\gamma_k{}^l n_k{}^l, \quad (35)$$

where

$$n_k{}^l\equiv\frac{\partial\epsilon_r}{\partial\omega_k{}^l}\frac{k^2\mid\phi_k{}^l\mid^2}{8\pi L^6}, \quad (36)$$

while (33) should generalize to

$$\frac{\partial F_0}{\partial t}+\frac{\partial\langle K\rangle}{\partial\mathbf{p}}\cdot\frac{\partial F_0}{\partial\mathbf{x}}-\frac{\partial\langle K\rangle}{\partial\mathbf{x}}\cdot\frac{\partial F_0}{\partial\mathbf{p}}=\frac{\partial}{\partial\mathbf{p}}\cdot\left(\mathsf{D}\cdot\frac{\partial F_0}{\partial\mathbf{p}}\right), \quad (37)$$

where $\langle\ \rangle$ denotes local space averaging, so that

$$\langle K\rangle=H_0+\Delta(\mathbf{p};\mathbf{x},t), \quad (38)$$

where Δ is given by Eq. (25). The length L of our Fourier box is taken to be much less than the inhomogeneity scale length.

To verify energy and momentum conservation, we must show that Eqs. (35) and (37) satisfy

$$\frac{\partial\langle\mathbf{G}\rangle}{\partial t}+\nabla\cdot\langle\mathsf{T}\rangle=0 \quad (39a)$$

and

$$\frac{\partial\langle W\rangle}{\partial t}+\nabla\cdot\langle\mathbf{S}\rangle=0, \quad (39b)$$

where the momentum density $\mathbf{G}$, stress tensor T, energy density W, and energy flux vector $\mathbf{S}$ are defined by

$$\mathbf{G}=\sum_s\int d^3p\,\mathbf{p}f(\mathbf{x},\mathbf{p},t), \quad (40a)$$

$$\mathsf{T}=\sum_s\int d^3p\,\frac{\partial H_0}{\partial\mathbf{p}}\mathbf{p}f(\mathbf{x},\mathbf{p},t)+\frac{E^2\mathsf{I}-2\mathbf{EE}}{8\pi}, \quad (40b)$$

$$W=\sum_s\int d^3p\,H_0 f(\mathbf{x},\mathbf{p},t)+[(\nabla\phi)^2/8\pi], \quad (40c)$$

$$\mathbf{S}=\sum_s\int d^3p\,\frac{\partial H_0}{\partial\mathbf{p}}H_0 f(\mathbf{x},\mathbf{p},t)+\frac{c\mathbf{E}\times\mathbf{B}}{4\pi}, \quad (40d)$$

where $H_0=p^2/2m$, $\mathbf{E}=-\nabla\phi$, and $\mathbf{B}$ is defined[14] by

$$c\nabla\times\mathbf{B}=4\pi\mathbf{j}+\frac{\partial\mathbf{E}}{\partial t},\qquad \nabla\cdot\mathbf{B}=0. \quad (41)$$

To calculate the average of Eq. (40) to $O(\lambda^2)$ we need the second-order contribution to $\langle f\rangle$, which we find from Eqs. (13), (14), and (16) to be given by

$$\langle f_2\rangle=\langle F_2\rangle+\frac{1}{2}\frac{\partial}{\partial \mathbf{p}}\cdot\left(\left\langle\frac{\partial S_1}{\partial \mathbf{x}}\frac{\partial S_1}{\partial \mathbf{x}}\right\rangle\cdot\frac{\partial F_0}{\partial \mathbf{p}}\right)$$
$$=-\frac{\partial}{\partial \mathbf{p}}\cdot\left(\sum_{k,l}4\pi e^2\frac{|\phi_k{}^l|^2}{8\pi L^6}\mathbf{kk}\frac{\partial}{\partial \omega_k{}^l}\frac{P}{\omega_k{}^l-\mathbf{k}\cdot\mathbf{p}/m}\cdot\frac{\partial F_0}{\partial \mathbf{p}}\right), \tag{42}$$

where we have inferred, from Kaufman's[4] work, that $\langle F_2\rangle$ acts to make $\langle f_2\rangle$ well defined by putting the principal part operator P in the correct place.

It is now a straightforward, if tedious, exercise to evaluate the averages of Eq. (40), and we find, using Eqs. (25), (28), and (30)

$$\langle \mathbf{G}\rangle=\sum_s\int d^3p\, F_0\mathbf{p}+\sum_{k,l}n_k{}^l\mathbf{k}, \tag{43a}$$

$$\langle \mathsf{T}\rangle=\sum_s\int d^3p\, F_0\frac{\partial\langle K\rangle}{\partial \mathbf{p}}\mathbf{p}+\sum_{k,l}n_k{}^l\frac{\partial \omega_k{}^l}{\partial \mathbf{k}}\mathbf{k}+\mathsf{I}\sum_{k,l}\frac{k^2|\phi_k{}^l|^2}{8\pi L^6}, \tag{43b}$$

$$\langle W\rangle=\sum_s\int d^3p\, F_0H_0+\sum_{k,l}n_k{}^l\omega_k{}^l, \tag{43c}$$

$$\langle \mathbf{S}\rangle=\sum_s\int d^3p\, F_0\frac{\partial\langle K\rangle}{\partial \mathbf{p}}\langle K\rangle+\sum_{k,l}n_k{}^l\frac{\partial \omega_k{}^l}{\partial \mathbf{k}}\omega_k{}^l, \tag{43d}$$

to $O(\lambda^2)$. Now using the microscopic conservation relations, which follow from Eqs. (32), (34), and (36),

$$\sum_{k,l}2\gamma_k{}^l n_k{}^l\mathbf{k}=\sum_s\int d^3p\,\mathsf{D}\cdot\frac{\partial F_0}{\partial \mathbf{p}},$$
$$\sum_{k,l}2\gamma_k{}^l n_k{}^l\omega_k{}^l=\sum_s\int d^3p\,\frac{\mathbf{p}}{m}\cdot\mathsf{D}\cdot\frac{\partial F_0}{\partial \mathbf{p}}, \tag{44}$$

and the identities, which use Eq. (30),

$$\sum_s\int d^3p\, F_0\frac{\partial\langle K\rangle}{\partial \mathbf{x}}+\sum_{k,l}n_k{}^l\frac{\partial \omega_k{}^l}{\partial \mathbf{x}}=\nabla\sum_{k,l}\frac{k^2|\phi_k{}^l|^2}{8\pi L^6},$$
$$\sum_s\int d^3p\, F_0\frac{\partial\langle K\rangle}{\partial t}+\sum_{k,l}n_k{}^l\frac{\partial \omega_k{}^l}{\partial t}=\frac{\partial}{\partial t}\sum_{k,l}\frac{k^2|\phi_k{}^l|^2}{8\pi L^6}, \tag{45}$$

we may show that, to $O(\lambda^2)$, the moments of Eqs. (35) and (37) are indeed consistent with Eqs. (39) and (43).

Note that we have assumed that $\sum_k$ is equivalent to an integral, enabling integration by parts. This continuum limit is really implicit in Eq. (35). Also note that the energy density comes out in the form

$$\langle W\rangle=\sum_s\int d^3p\, F_0\langle K\rangle+\sum_k n_k{}^l\omega_k{}^l-\sum_k(k^2|\phi_k{}^l|^2/8\pi L^6), \tag{46}$$

which is actually the form expected from application of Noether's theorem to the averaged Lagrangian density of the plasma.[15] However, use of the identity

$$\sum_s\int d^3p\, F_0\Delta=\sum_{k,l}(k^2|\phi_k{}^l|^2/8\pi L^6)$$

shows that Eqs. (43c) and (46) are equivalent.

If we had used H_0 rather than $\langle K\rangle$ in Eq. (37), we would not have satisfied the conservation conditions, showing that the prescription given in Ref. 1 for generalizing the quasilinear diffusion equation is only correct if H is renormalized. The physical interpretation of the renormalization energy Δ is clear, it being simply a generalization of the "high-frequency potential" used in the theory of rf confinement.[16] We have thus shown how to combine both radiation pressure and resonant diffusion in one equation.

V. DISCUSSION

The approach of this paper has been predicated on the assumption that there is a fundamental distinction between resonant and nonresonant interaction, the resonant interaction being essentially stochastic and nonresonant interaction being adiabatic. This idea is strengthened somewhat by recent work on particles accelerated by periodic forces.[17] However, it is not clear how the renormalization of the nonresonant interaction introduced in this paper is related to the renormalization of the resonant interaction introduced by Dupree.[18] Unifying the two approaches must be left to future work.

It is clear that our transformation has application in plasma kinetic theory as well as quasilinear theory. It is easily seen that the fluctuating field in the plasma is simply the superposition of the dielectrically screened fields of fictitious particles (quasiparticles) moving with the oscillation centers. Thus, the oscillation center description formalizes Hubbard's interpretation of the dressed test particle picture.[19,20] Probably the most important application of our method is in the theory of momentum and heat transport across magnetic fields, where it has been suggested[21] that the long mean free path of waves compared with particles enhances their effect. Examination of Eq. (43b) shows that some care must be taken in order to extract the correct viscous effect, and generalization of our transformation to the case of strong external magnetic field would provide a means for treating this problem.

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