

Bernstein-Greene-Kruskal Modes in a Three-Dimensional Plasma

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Bernstein-Greene-Kruskal modes in a three-dimensional (3D) unmagnetized plasma are constructed. It is shown that 3D solutions that depend only on energy do not exist. However, 3D solutions that depend on energy and additional constants of motion (such as angular momentum) do exist. Exact analytical as well as numerical solutions are constructed assuming spherical symmetry, and their properties are contrasted with those of 1D solutions. Possible extensions to solutions with cylindrical symmetry with or without a finite magnetic guide field are discussed.

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A Bernstein-Greene-Kruskal (BGK) mode is an exact nonlinear solution of the Vlasov-Poisson system of equations. The original BGK solutions, obtained by Bernstein, Greene, and Kruskal in a classic paper [1], represent exact nonlinear plasma waves that do not experience any Landau damping in a one-dimensional (1D) unmagnetized plasma. If such a solution is stable, it can survive a long time in a weakly collisional plasma. In the small-amplitude limit, BGK solutions can be shown to reduce to singular Case–Van Kampen modes [2,3], which are free of Landau damping. Although Case–Van Kampen modes are singular, the exact BGK wave solutions from which they are derived by a small-amplitude expansion are perfectly well-behaved.

BGK modes have been observed experimentally in quasinneutral [4,5] as well as trapped pure electron plasmas [6]. Numerous space-based observations of solitary waves or phase space hole structures in the magnetosphere [7–11] and the solar wind [12], which are seen to persist as nonlinear structures, have been interpreted as BGK modes [13,14]. Numerical simulations seem to suggest that stable 1D nonlinear modes of the BGK type can be formed dynamically via a two-stream instability [15]. See Ref. [16] for a recent review of 1D BGK theory.

Surprisingly, there appear to have been very few theoretical attempts to construct exact BGK solutions analytically in two and three dimensions (2D and 3D). In recent years, there has been renewed interest in 3D BGK modes, motivated mainly by distinct 3D features of solitary wave structures in space-based observations that cannot be explained by 1D theory [8–10]. For instance, it has been found that electrostatic solitary waves observed in the auroral ionosphere are characterized by electric field components E_{\perp} perpendicular to the background magnetic field with magnitudes comparable to the parallel component E_{\parallel} [8]. Recently, 3D BGK solutions have been constructed under the assumption of an infinitely strong background magnetic field [17–19] or a sufficiently strong magnetic field so that one can use some form of the gyrokinetic or drift-kinetic equation [20,21]. The infinitely strong field constrains charged particles to move along magnetic field lines like beads on a wire and, thus, effec-

tively reduces the 3D problem to a 1D problem [17–19]. Despite such a strong assumption, this theory has generated interest for further development and comparison with observation data [21]. However, efforts to relax this assumption to magnetic fields of finite strength have not produced exact results so far, and it has been conjectured that a 3D BGK solution for zero magnetic field does not exist [17].

In this Letter, we develop a simple but rigorous approach in constructing BGK solutions in 3D. We demonstrate that 2D and 3D solutions in which the particle distribution function f is assumed to depend only on energy do *not* exist, consistent with the conjecture in Ref. [17]. However, we show that, if we allow f to have dependencies on additional constants of motion (such as the angular momentum), BGK solutions do exist in 3D. We prove this by actually constructing a class of exact 3D BGK solutions in an unmagnetized, spherically symmetric plasma. We then discuss briefly how our approach may be extended to obtain solutions with cylindrical symmetry, with or without a magnetic field of finite strength.

We start with the Vlasov equation for each species s of the plasma with mass m_s , charge q_s , and distribution function $f_s(\mathbf{r}, \mathbf{v}, t)$ as a function of space, time, and velocity. We consider nonlinear electrostatic waves, so that the electric field can be represented as the gradient of a scalar electric potential $\psi(\mathbf{r})$, that is, $\mathbf{E} = -\nabla\psi$. The relevant Vlasov-Poisson system of equations describing stationary electrostatic waves is

$$\mathbf{v} \cdot \frac{\partial f_s}{\partial \mathbf{r}} - \frac{q_s}{m_s} \nabla\psi \cdot \frac{\partial f_s}{\partial \mathbf{v}} = 0, \quad (1)$$

$$\nabla^2\psi = -4\pi \sum_s q_s \int d\mathbf{v} f_s. \quad (2)$$

For simplicity, we assume that the much more massive ions form a uniform background with constant density n_0 and solve only for the electron distribution. Using the normalizations $\mathbf{v} \rightarrow v_e \mathbf{v}$, $\mathbf{r} \rightarrow \lambda \mathbf{r}$, $\psi \rightarrow 4\pi n_0 e \lambda^2 \psi$, and $f_e \rightarrow n_0 f_e / v_e^3$, where v_e is the electron thermal velocity and $\lambda = v_e / \omega_{pe}$ is the electron Debye length, (1) and (2) become

$$\mathbf{v} \cdot \frac{\partial f}{\partial \mathbf{r}} + \nabla \psi \cdot \frac{\partial f}{\partial \mathbf{v}} = 0, \quad (3)$$

$$\nabla^2 \psi = \int d\mathbf{v} f - 1, \quad (4)$$

where we have dropped the subscript e on the electron distribution function. It is easy to see that (3) is satisfied exactly by a distribution function of the form $f = f(w)$, where $w = v^2/2 - \psi(\mathbf{r})$ is the energy. However, in order to be a truly self-consistent solution of the Vlasov-Poisson system, a distribution function of the form $f = f(w)$ must also satisfy (4). We now present a simple proof that no such solutions exist in 2D and 3D.

In 2D, (3) and (4) give $n \equiv \nabla^2 \psi = 2\pi \int_{-\psi}^{\infty} dw f(w) - 1$. It follows that $dn/d\psi = 2\pi f(-\psi) \geq 0$. The derivation of a similar relation in 3D is technically a bit more involved but straightforward. We first define a 1D reduced distribution

$$g\left(\frac{v_z^2}{2} - \psi\right) = 2\pi \int_0^{\infty} v_{\perp} dv_{\perp} f\left(\frac{v_{\perp}^2}{2} + \frac{v_z^2}{2} - \psi\right), \quad (5)$$

where v_{\perp} is the velocity component perpendicular to the z direction. It follows from definition (5) that $g(w) = 2\pi \int_w^{\infty} d\xi f(\xi)$, and, hence, $g'(w) = -2\pi f(w) \leq 0$. Using these properties of g , we can write

$$n \equiv \nabla^2 \psi = \int_{-\infty}^{\infty} dv_z g\left(\frac{v_z^2}{2} - \psi\right) - 1 \quad (6)$$

so that

$$\frac{dn}{d\psi} = - \int_{-\infty}^{\infty} dv_z g'\left(\frac{v_z^2}{2} - \psi\right) \geq 0. \quad (7)$$

The inequality $dn/d\psi \geq 0$, which holds in 2D as well as 3D, leads to the conclusion that it would be impossible to obtain a physical and localized solution that satisfies the boundary conditions $\psi \rightarrow 0$ and $n \rightarrow 0$ as $|\mathbf{r}| \rightarrow \infty$. The reasons are as follows. For such a localized solution, ψ must have either a local maximum that is positive or a local minimum that is negative at a certain point in space. However, since $n = \nabla^2 \psi$ is negative (positive) in these two cases, then according to the inequality $dn/d\psi \geq 0$, n must be increasingly negative (positive) as ψ decreases (increases) from its local maximum (minimum). It would thus be impossible to satisfy the boundary conditions $\psi \rightarrow 0$ and $n \rightarrow 0$ as $|\mathbf{r}| \rightarrow \infty$ simultaneously, required for a localized solution.

We now demonstrate that it is possible to construct 3D solutions if we relax the restriction that the distribution function depends only on energy. Let us consider the case in which the electrostatic potential is spherically symmetric, that is, $\psi = \psi(r)$. The distribution function will then have the form $f = f(r, v_r, v_{\perp})$, where $v_{\perp}^2 = v_{\theta}^2 + v_{\phi}^2$, and (r, θ, ϕ) are the usual spherical coordinates. With these simplifications and coordinate transformations, (3) and (4) become [22,23]

$$v_r \frac{\partial f}{\partial r} + \left(\frac{d\psi}{dr} + \frac{v_{\perp}^2}{r}\right) \frac{\partial f}{\partial v_r} - \frac{v_r v_{\perp}}{r} \frac{\partial f}{\partial v_{\perp}} = 0, \quad (8)$$

$$\frac{1}{r} \frac{d^2(r\psi)}{dr^2} = \pi \int_0^{\infty} d(v_{\perp}^2) \int_{-\infty}^{\infty} dv_r f - 1. \quad (9)$$

We first look at a simple, exact solution of (8) and (9) that illustrates the essential physics of such solutions. It can be shown by direct substitution that the distribution function

$$f = g(r) \delta(v_r) \delta\left(v_{\perp}^2 + r \frac{d\psi}{dr}\right) \quad (10)$$

satisfies (8), where $g(r)$ is an arbitrary positive function. We assume that $d\psi/dr \leq 0$. As shown in Fig. 1, this solution represents a situation in which electrons move in circular orbits around a spherically symmetric positive electrostatic potential with the electric field always pointing radially outwards and the attractive electric force on any electron balanced by its centripetal acceleration. Equation (9) then requires that

$$\frac{1}{r} \frac{d^2(r\psi)}{dr^2} = \pi g(r) - 1. \quad (11)$$

There are many solutions $\psi(r)$ that result in a positive $g(r)$, with $d\psi/dr \leq 0$. As an example, we consider $\psi(r) = \psi_0 \exp(-r^2/\delta^2)$. Then the requirement that $g(r)$ be positive yields the condition $\delta^2 > 6\psi_0$, which restricts the width-amplitude relation of the electrostatic potential. For a more detailed discussion of other such width-amplitude inequalities that are relevant to observations in space, the reader is referred to Ref. [14].

The simple example discussed above shows that, if we can arrange electrons to circulate around the potential and to allow dependence on angular momentum, it becomes possible to construct 3D solutions. We now consider a more general solution of (8), $f = f(w, l)$, where $l = v_{\perp} r$ is the angular momentum. [It can be shown that the special solution in Eq. (10) can be put into this form implicitly,

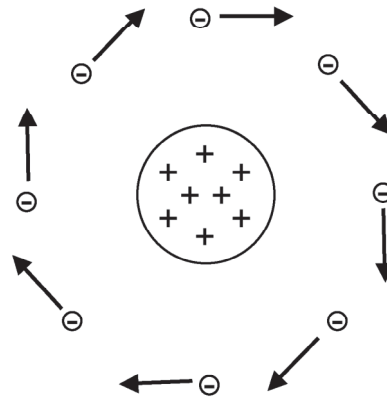


FIG. 1. Schematic illustration of the role of electron angular momentum in realizing 3D BGK solutions. The radial electric field provides the centripetal force necessary to sustain the circular electron motion.

after taking into account the delta functions.] The next step is to find specific solutions of this form that also satisfy (9). For simplicity, we consider solutions of the separable form $f(w, l) = (2\pi)^{-3/2} \exp(-w)f_1(l)$. Since we are looking for a solution localized in r , we impose the boundary condition that, as $\psi \rightarrow 0$, the distribution function must tend to a Maxwellian, that is, $f \rightarrow (2\pi)^{-3/2} \exp(-w)$. Hence, $f_1 \rightarrow 1$ as $r \rightarrow \infty$. For specificity, we choose

$$f_1(v_{\perp} r) = 1 - (1 - h_0) \exp(-v_{\perp}^2 r^2/x_0^2), \quad (12)$$

with two real parameters h_0 and x_0 . We note that $f_1(0) = h_0 \geq 0$, and $f_1(\infty) = 1$. Equation (9) then becomes

$$\frac{1}{r} \frac{d^2(r\psi)}{dr^2} = e^{\psi} h(r) - 1, \quad (13)$$

where we obtain by direct integration $h(r) = (h_0 + 2r^2/x_0^2)/(1 + 2r^2/x_0^2)$. Equation (13) is a second-order nonlinear ordinary differential equation for $\psi(r)$, to be solved subject to the boundary conditions $\psi(r \rightarrow \infty) \rightarrow 0$, $\psi(r=0) = \psi_0$, and $\psi'(r=0) = 0$. For the special case $h_0 = 1$, it follows that $f_1 = h = 1$ for all r . Then the distribution function depends only on energy, in which case, by our discussion above, there is no nontrivial solution. That this is so can also be seen from (13), which yields only the trivial solution $\psi(r) \equiv 0$ if $h = 1$. If this were not so, then for any nonzero positive (negative) value of ψ_0 , $\psi(r \rightarrow \infty)$ will tend to positive (negative) infinity, which violates the boundary condition as $r \rightarrow \infty$.

We now demonstrate, by an extension of the argument given above, that when $h_0 \neq 1$, a well-behaved solution must exist for a certain ψ_0 . First, we note that if $0 \leq h_0 < 1$, for $\psi_0 > \ln(1/h_0)$, $\psi(r \rightarrow \infty) \rightarrow \infty$. Second, for sufficiently small ψ_0 , including all negative ψ_0 , ψ will become negative at some point, leading to $\psi(r \rightarrow \infty) \rightarrow -\infty$. Therefore, we see that there must exist a value of ψ_0 in between these two cases that satisfies the required boundary condition $\psi(r \rightarrow \infty) \rightarrow 0$. Stated differently, Eq. (13), subject to the boundary condition $\psi(r \rightarrow \infty) \rightarrow 0$, is an eigenvalue problem for ψ_0 that can be solved numerically.

Figure 2(a) shows the solution $\psi(r)$ for the case with $h_0 = 0.9$ and $x_0 = 1$. The same solution is plotted on log-log scale in Fig. 2(b), showing the asymptotic behavior of $\psi(r \rightarrow \infty) \rightarrow \psi_{\infty}/r^2$ for a constant ψ_{∞} . In fact, by directly substituting this asymptotic form into (13), the value of ψ_{∞} is found to be $\psi_{\infty} = x_0^2(1 - h_0)/2$. This also means that the electric field has the asymptotic behavior $\mathbf{E}(r \rightarrow \infty) \rightarrow 2\psi_{\infty} \mathbf{r}/r^4$. Since the electric field falls off faster than $1/r^2$, the global solution is asymptotically charge-neutral. We plot the radial electric field in Fig. 2(c) and the normalized charge density $\rho_q(r) \equiv 1 - e^{\psi} h(r)$ in Fig. 2(d). Careful inspection shows a long tail for ρ_q moving from positive to negative values at around $r \sim 2.7$.

For the case $h_0 > 1$, we can demonstrate similarly that there must exist a negative ψ_0 that satisfies the required boundary condition $\psi(r \rightarrow \infty) \rightarrow 0$. Figure 3 shows the solutions when $h_0 = 1.1$ and $x_0 = 1$, which are very simi-

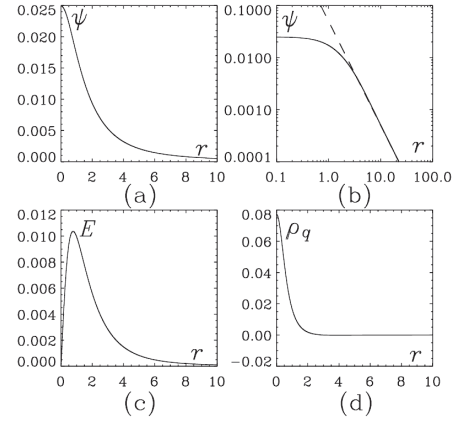


FIG. 2. (a) Numerical solution $\psi(r)$ for the case with $h_0 = 0.9$ and $x_0 = 1$. (b) The same solution in log-log plot. The dashed line shows the relation $\psi(r \rightarrow \infty) \rightarrow \psi_{\infty}/r^2$. (c) Radial electric field. (d) Normalized charge density $\rho_q(r) \equiv 1 - e^{\psi} h(r)$.

lar to the solutions represented in Fig. 2 except that their signs are reversed. This is an interesting case because a negative potential is supported in 3D entirely by electron dynamics. It would be impossible to realize such a solution in 1D, since all electrons in such a potential will have passing trajectories that will tend to infinity, where the distribution must be a Maxwellian. This will not support a localized BGK solution with negative potential. The only way a localized BGK solution with negative potential can be supported in 1D is by taking into account the dynamics of trapped ions.

It is interesting to determine the relation between the potential ψ_0 and the characteristic length x_0 . Figure 4 shows the plots of ψ_0 vs x_0 for various values of h_0 . For large x_0 , ψ_0 simply tends to $\ln(1/h_0)$. As $x_0 \rightarrow 0$, we see that $\psi_0 \rightarrow 0$. In this limit, the functional dependence is very similar to that of ψ_{∞} , except that it has a larger numerical coefficient, that is, $\psi_0 \sim 8x_0^2(1 - h_0)$. We note that even in this limit the solution itself as well as the potential ψ cannot be recovered by solving the linearized

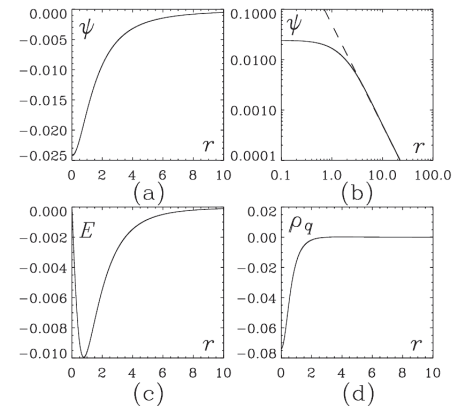


FIG. 3. Same as in Fig. 1 for the case $h_0 = 1.1$ and $x_0 = 1$. The profiles are nearly identical as those in Fig. 2 except that the signs are reversed.

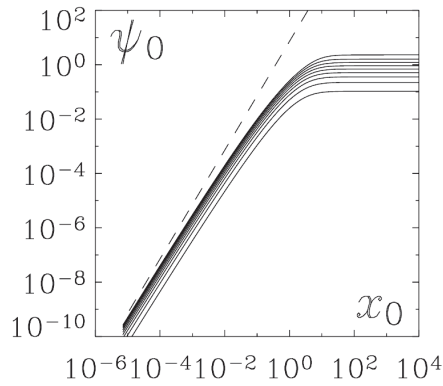


FIG. 4. Plots of ψ_0 versus x_0 for various values of h_0 from 0.1 (top curves) to 0.9 (bottom curve) in increments of 0.1. The dashed line is $\psi_0 = 8x_0^2(1 - h_0)$ for $h_0 = 0.1$.

Vlasov equation, since the velocity gradient of f is large around $v_\perp \rightarrow 0$. In other words, this is a truly nonlinear solution even in the small-amplitude limit.

We remark that the width-amplitude inequality based on these solutions is very similar to that given in Ref. [14], which has been shown to be consistent with observations. The feature that the electrostatic potential of these 3D solutions is typically single-humped in all directions is consistent with observations [13] that solitary waves in the auroral ionosphere are bipolar in the parallel electric field E_\parallel and unipolar in both components of perpendicular electric field E_\perp . It is interesting that these features are realized even without including the background magnetic field in our solutions.

We should emphasize that the solutions presented here are a few among many, chosen to illustrate the principles of our approach. One major extension of our approach will be the inclusion of possible piecewise distribution functions distinguishing trapped and passing electrons. Such distinction is not made in the solutions presented here, nor are they necessary, because we are able to describe the whole distribution function with a single continuous and smooth function. This should be contrasted with the 1D case, where the distinction between trapped and passing particles must be made in order to construct a solution.

Finally, we discuss briefly the possibility of relaxing the above spherically symmetric case to the cylindrically symmetric case with $\psi = \psi(\rho, z)$, $f = f(\rho, z, v_\rho, v_\phi, v_z)$. For generality, we also include a uniform background magnetic field $\mathbf{B} = B\hat{z}$. The Vlasov equation can then be solved by a distribution function of the form $f = f(w, 2\rho v_\phi - B\rho^2)$. To find a BGK solution, one then needs to solve the Poisson equation self-consistently. The result is a partial differential equation for ψ in two variables instead of the ordinary differential equation in (13), and, thus, substantially more analysis is needed to discuss its solution. Therefore, we will defer that to a later publication.

In summary, we have shown that a 3D BGK solution can be constructed in an unmagnetized plasma if and only if the

distribution function is not only dependent on energy but also on other constants of motion such as the angular momentum. We have constructed a class of such solutions and analyzed their functional form, as well as their amplitude-width relation. We have not considered the dynamical accessibility and stability of such solutions, which is likely to require large-scale numerical simulations. This is left to future investigation.

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- [1] I. B. Bernstein, J. M. Greene, and M. D. Kruskal, *Phys. Rev.* **108**, 546 (1957).
 - [2] N. G. Van Kampen, *Physica (Amsterdam)* **21**, 949 (1955).
 - [3] K. M. Case, *Ann. Phys. (N.Y.)* **7**, 349 (1959).
 - [4] J. P. Lynov, P. Michelsen, H. L. Pecseli, J. J. Rasmussen, K. S. Saeki, and V. A. Turikov, *Phys. Scr.* **20**, 328 (1979).
 - [5] K. Saeki, P. Michelsen, H. L. Pecseli, and J. J. Rasmussen, *Phys. Rev. Lett.* **42**, 501 (1979).
 - [6] J. R. Danielson, F. Andereg, and C. F. Driscoll, *Phys. Rev. Lett.* **92**, 245003 (2004).
 - [7] M. Temerin, K. Cerny, W. Lotko, and F. S. Mozer, *Phys. Rev. Lett.* **48**, 1175 (1982).
 - [8] H. Matsumoto, H. Kojima, T. Miyatake, Y. Omura, M. Okada, I. Nagano, and M. Tsutsui, *Geophys. Res. Lett.* **21**, 2915 (1994).
 - [9] R. E. Ergun, C. W. Carlson, J. P. McFadden, F. S. Mozer, L. Muschietti, I. Roth, and R. J. Strangeway, *Phys. Rev. Lett.* **81**, 826 (1998).
 - [10] J. Franz, P. M. Kintner, and J. S. Pickett, *Geophys. Res. Lett.* **25**, 1277 (1998).
 - [11] C. A. Cattell, J. Dombeck, J. R. Wygant, M. K. Hudson, F. S. Mozer, M. A. Temerin, W. K. Peterson, C. A. Kletzing, and C. T. Russell, *Geophys. Res. Lett.* **26**, 425 (1999).
 - [12] A. Mangeney, C. Salem, C. Lacombe, J.-L. Bougeret, C. Perche, R. Manning, P. J. Kellogg, K. Goetz, S. J. Monson, and J.-M. Bosqued, *Ann. Geophys.* **17**, 307 (1999).
 - [13] L. Muschietti, R. E. Ergun, I. Roth, and C. W. Carlson, *Geophys. Res. Lett.* **26**, 1093 (1999).
 - [14] L.-J. Chen, J. Pickett, P. Kintner, J. Franz, and D. Gurnett, *J. Geophys. Res.* **110**, A09211 (2005).
 - [15] K. V. Roberts and H. L. Berk, *Phys. Rev. Lett.* **19**, 297 (1967).
 - [16] H. Schamel, *Phys. Plasmas* **7**, 4831 (2000).
 - [17] L.-J. Chen, Ph.D. thesis, University of Washington, 2002.
 - [18] L.-J. Chen and G. K. Parks, *Geophys. Res. Lett.* **29**, 1331 (2002).
 - [19] L.-J. Chen, D. J. Thouless, and J.-M. Tang, *Phys. Rev. E* **69**, 055401(R) (2004).
 - [20] D. Jovanovic and P. K. Shukla, *Phys. Rev. Lett.* **84**, 4373 (2000).
 - [21] V. L. Krasovsky, H. Matsumoto, and Y. Omura, *J. Geophys. Res.* **109**, A04217 (2004); *Nonlinear Processes Geophys.* **11**, 313 (2004).
 - [22] J. Binney, and S. Tremaine, *Galactic Dynamics* (Princeton, Princeton, 1987).
 - [23] R. Fusco-Femiano and N. Menci, *Astrophys. J.* **498**, 95 (1998).